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Berezinskii–Kosterlitz–Thouless transition

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Phase transition in terms of order parameter

-Order parameter in Ising model: magnetisation M

$$H_N = -B\sum_{i=1}^{N} S_i - J\sum_{\langle i,j\rangle} S_i S_j \quad (S_i \in \{-1,+1\}) \qquad f = -\lim_{N\to\infty}\frac{1}{N}\frac{1}{\beta}\log \mathrm{Tr} e^{-\beta H_N}$$

$$M = -\frac{\partial f}{\partial B} = \lim_{N\to\infty}\frac{1}{N}\left\langle \sum_{i=1}^{N} S_i \right\rangle \qquad M_s = \lim_{B\to 0^{\pm}} M(B)$$

-Energy-Entropy argument for phase transition

-High T: F is dominated by TS, maximising S leads to minimal F $\Rightarrow M_s(T \gg 0) = 0$

-Low T: F is dominated by E, minimising E leads to minimal F

-T=0: minimal F=E for $S_i = \begin{cases} +1, & B \to 0^+ \\ -1, & B \to 0^- \end{cases} \Rightarrow M_s(T=0) = \pm 1$

-Possibility of phase transition at finite temperature

Phase transition and dimensionality 1

-No phase transition for T>0 in 1D Ising model

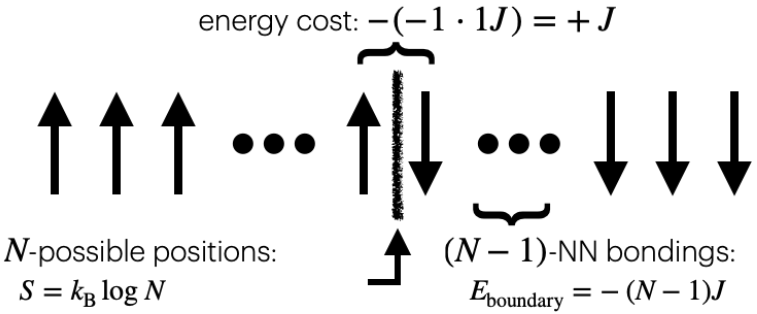
$$F_{\text{onephase}} = -NJ$$

$$F_{\text{boundary}} = -(N-1)J + J - TS = -NJ + 2J - k_{\text{B}}T \log N$$

$$\Delta F = F_{\text{boundary}} - F_{\text{onephase}} = 2J - k_{\text{B}}T \log N$$

-F is minimised by increasing # boundaries/domains

-Thermal fluctuations prevents magnetisation: No phase transition



Phase transition and dimensionality 2

-Phase transition in 2D Ising model

-Domain boundary deformation: $S \leq k_B \log(z-1)^b$

$$\Delta F = b[2J - k_B T \log(z-1)]$$

$$T_c = \frac{2J}{k_B \log(z-1)}$$

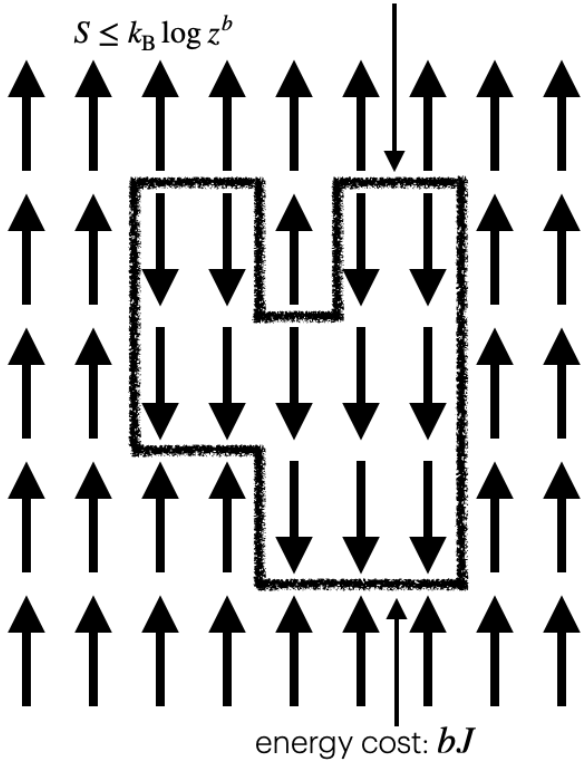
-Spontaneous Symmetry Breaking (SSB) in 2D Ising model

-Phase transition depends on dimensionality

b: # NN-bonds (length of boundary)

z: coordination number of lattice

z^b-maximum possible configurations:



Phase transition and symmetry

-Extension from Ising model to O(n) model: continuous symmetry

$$H = -J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j \qquad \vec{S}_i = (S_i^{(1)}, S_i^{(2)}, \ldots, S_i^{(n)}) \in \mathbb{R}^n$$

-Heisenberg model with n=3, XY model with n=2

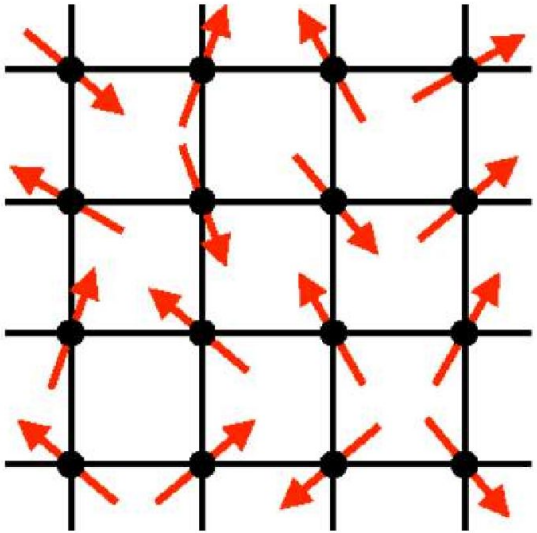
-No SSB for T>0 in XY model (Mermin–Wagner–Hohenberg)

-Phase transition depends on symmetry

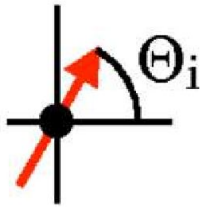
Two Dimensional XY-model

On the lines of Henrik Jeldtoft Jensen, Department of Mathematics, Imperial College

$$H = E_0 + \frac{J}{2}\int d\mathbf{r}(\nabla\theta)^2. \qquad \text{os}(\theta_i - \theta_j).$$



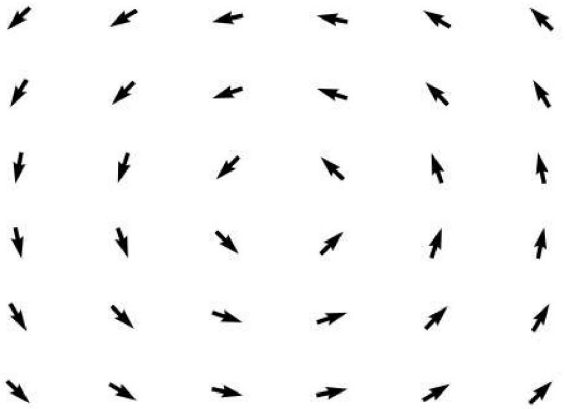
- No external magnetisation term



rotor field

Anders W. Sandvik, Boston University, 2010

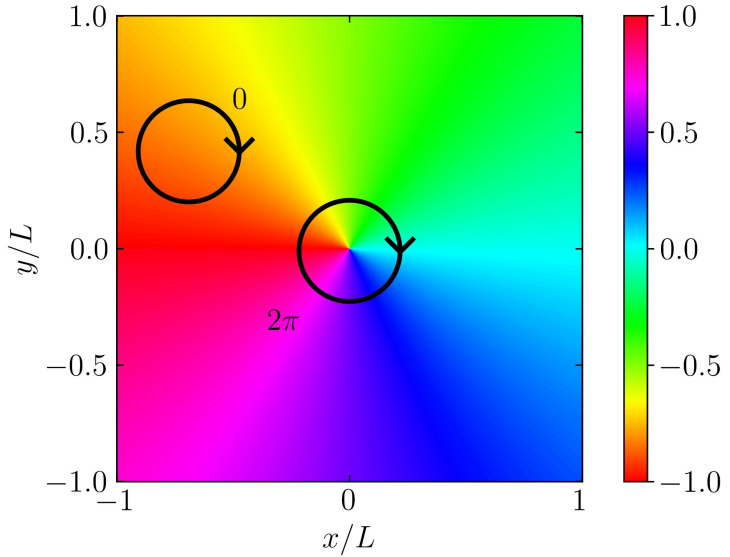
Minimum energy $\theta(r)$ configurations



$$\frac{\delta H}{\delta \theta(\mathbf{r})} = 0 \;\Rightarrow\; \nabla^2 \theta(\mathbf{r}) = 0.$$

- Ground state: θ is constant
- T>0 vortices! (phase winding of *2 pi* around phase singularity)

$$E_{vor} - E_0 \;=\; \pi n^2 J \ln(\frac{L}{a}).$$



“Bound vortex pairs”

- At much larger scales wrt to the distance of the V-AV pair, the distortion of the director field is ~0 !
- Interaction energy is LOGARITHMIC in the DISTANCE of two vortices *R*

$$E_{2vor}(R) = 2E_c + E_1 \ln(R/a).$$

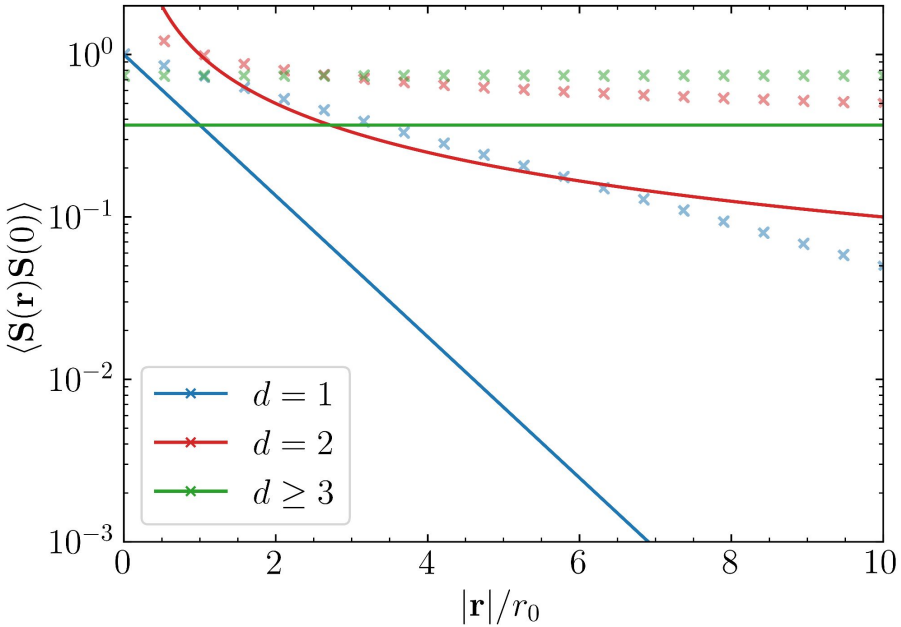
(*a* is a lower cutoff below which the density is zero: grid size in spin models, healing length in Bose gases …)

→ **quasi-long-range-order**

THE SHAPE of the INTERACTIONS is essential for the existence of a non-conventional phase transition in 2 DIMENSIONS.

Spins correlation function

$$\langle \mathbf{S}(\mathbf{r})\mathbf{S}(0)\rangle \simeq \begin{cases} e^{-\text{const.}T} & \text{for } d > 2 \\ \left(\frac{r}{L}\right)^{-\eta} & \text{for } d = 2 \\ \exp(-\frac{T}{2Ja}r) & \text{for } d = 1. \end{cases} \qquad \eta = T/2\pi J$$



d=2: Correlations' decay is polynomial (at T>0), constant at T=0

Same in presence of bound V-AV pairs.

BUT

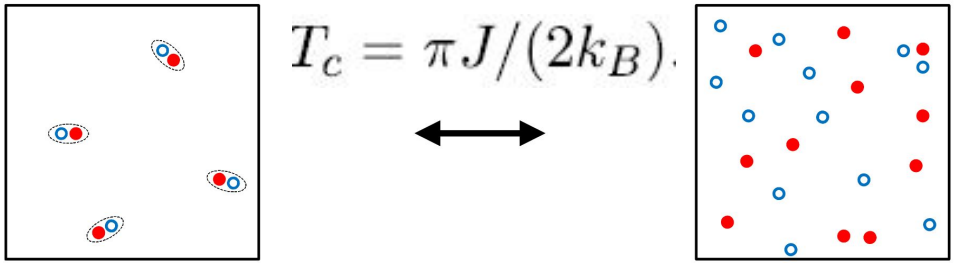
above T_KBT, "vortex gas" destroys log decay

Vortex binding

The nucleation of “free”, unbound vortices, is favoured above T_BKT…

$$F = E_0 + (\pi J - 2k_B T)\ln(L/a).$$

… and strongly disfavoured below. What happens: transition to two-vortex bound states.



Dalibard, Trieste

Spin-wave stiffness

$$F(\mathbf{v}_{ex}) - F(0) = \frac{1}{2} V \rho_s^R v_{ex}^2.$$

ϱ_s^R is J at zero temperature. v_ex is the spin twisting (gradient). Renormalized stiffness in terms of the correlation function of the Fourier transform of the vortex density function n(r):

$$\rho_s^R = \rho_s - \frac{1}{2}\frac{\rho_s^2}{T} \lim_{k \to 0} \frac{\langle \hat{n}(\mathbf{k}) \hat{n}(-\mathbf{k}) \rangle_)}{k^2}.$$

$$n(\mathbf{r}) = \sum_\alpha n_\alpha \delta(\mathbf{r} - \mathbf{r}_\alpha),$$

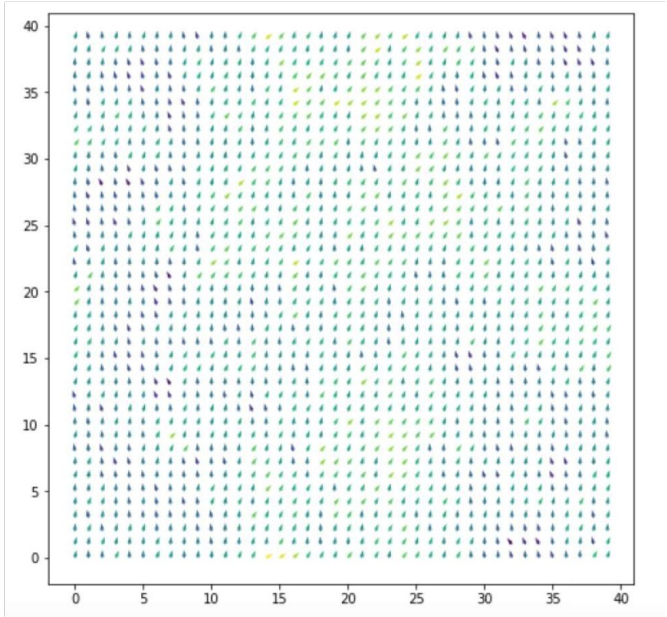
Spin-wave stiffness

$$\rho_s^R = \begin{cases} \rho_s^R(T_{KT}^-)[1 + const.(T_{KT} - T)^{1/2}] & \text{for } T < T_{KT} \\ 0 & \text{for } T > T_{KT} \end{cases} .$$

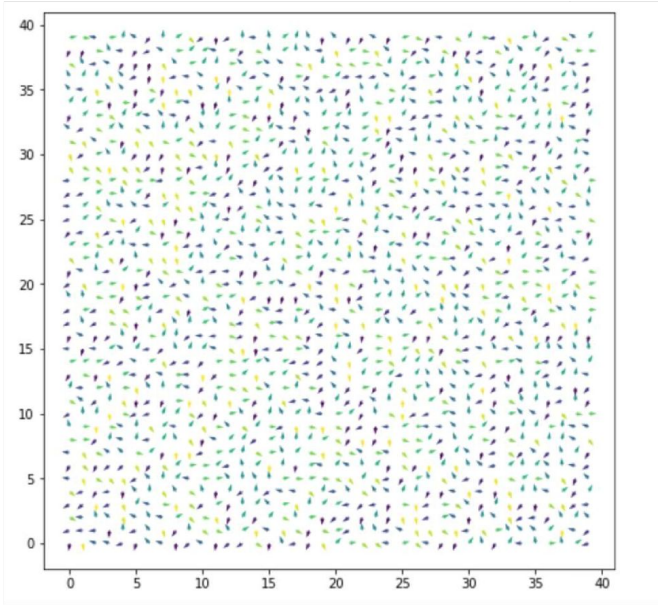
$$\rho_s^R(T_{KT}^-)/T_{KT} = 2/\pi$$

Universal jump for any BKT transition

Quasi long-range order



(a) Low temperature regime.



(b) High temperature regime.

Figure 7: Spin configuration.

Meabh Allen, MSc Thesis, Imperial College London (2021)

Note

For 2D topological charges both energy and entropy have a log dependency on the system's scale.

No symmetry breaking, but: at the transition, the correlations' decay is only algebraic.

Berezinskii, Kosterlitz, Thouless early 70s. 2016 Nobel Prize "for theoretical discoveries of topological phase transitions and topological phases of matter"

Analogy between XY-Model and 2D Bose Gas

2D GPE Energy functional $E[\psi] = \int_A d^2 r \frac{\hbar^2}{2m} |\nabla \psi|^2 + \frac{\hbar^2}{2m} \tilde{g} |\psi|^4$

Decompose into amplitude-phase representation $\psi(\mathbf{r}) = \sqrt{\varrho(\mathbf{r})} e^{i\theta(\mathbf{r})}$

$$E[\psi] \approx E_{in,0} + \frac{\hbar^2}{2m} \rho_0 \int_A d^2 r (\nabla \theta)^2 + \delta E[\delta \rho]$$

Background energy

Bosonic density

Amplitude fluctuations

Equation of motion for the phase $\nabla^2 \theta(\mathbf{r}) = 0$

Same MF equation of motion as XY model angle

Effects on correlation function

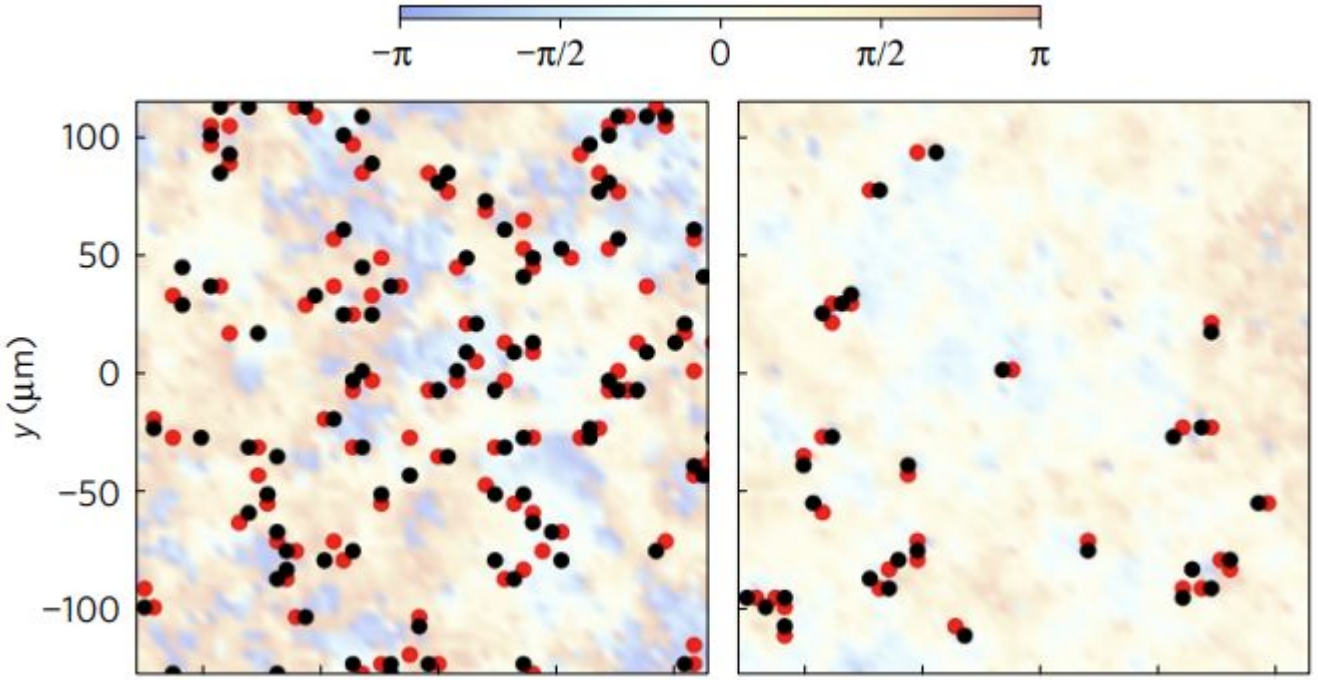
- Analogous to XY-Model: Vortex solutions possible
- Single Vortex has divergent energy with system size, positive or negative
- Vortex pair has finite energy (recall XY spins)

➢ Below temperature T_0 creation of pairs is favoured

  - Below certain temperature T0 pair creation is favoured
    - No change in correlation since phases cancel
  - Above T0 free vortex creation is energetically favoured (See talk Sá de Melo)
    - Correlations are changed to exponential decay due to free vortex gas

➢ Below T_0 supefluidity is preserved, above abruptly destroyed

Numerical evaluation of critical temperature



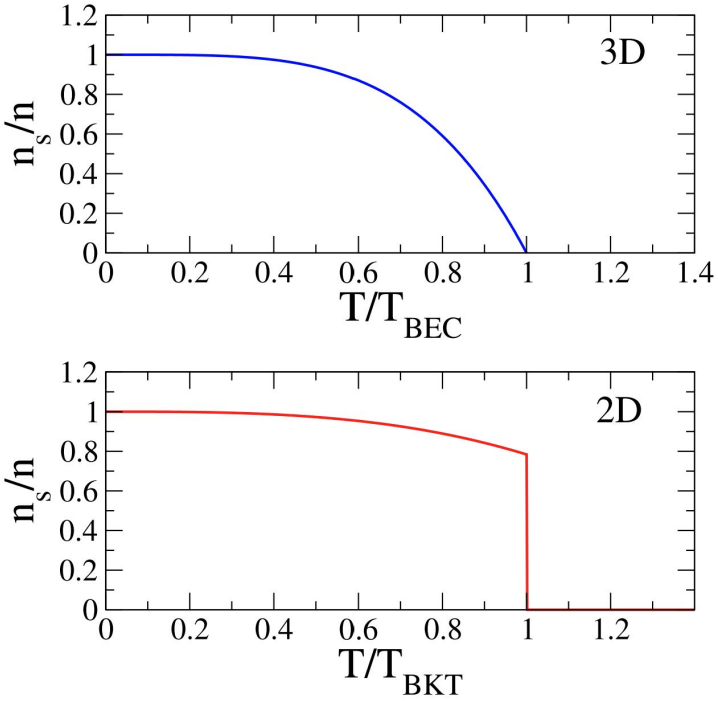
Daniele Sanvitto et. al., *Nature Materials* **17**, 145-151 (2018)

Superfluid density

- Nelson criterion for SF density
  - Self-consistent dependence on T_KT
- Universal functional shape of BKT transition

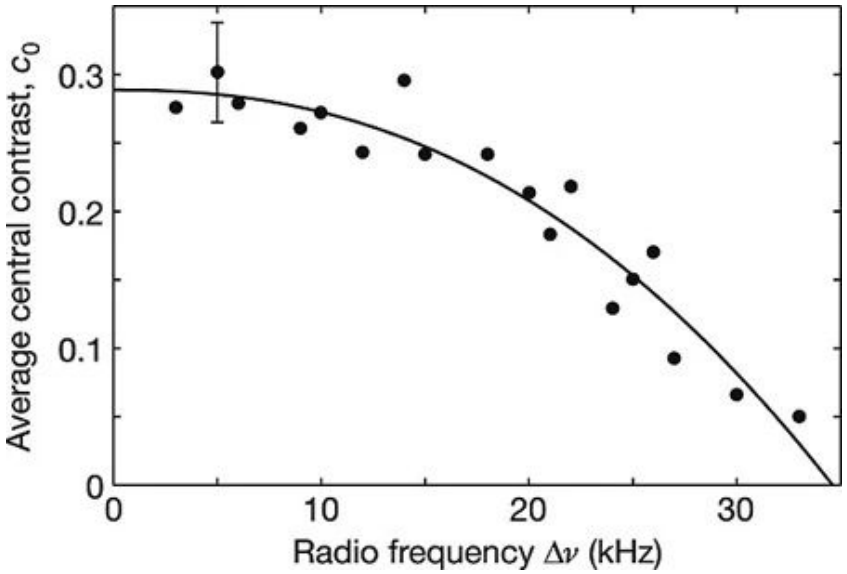
$$n_s(T_{BKT}) = \frac{2}{\pi} T_{BKT}$$

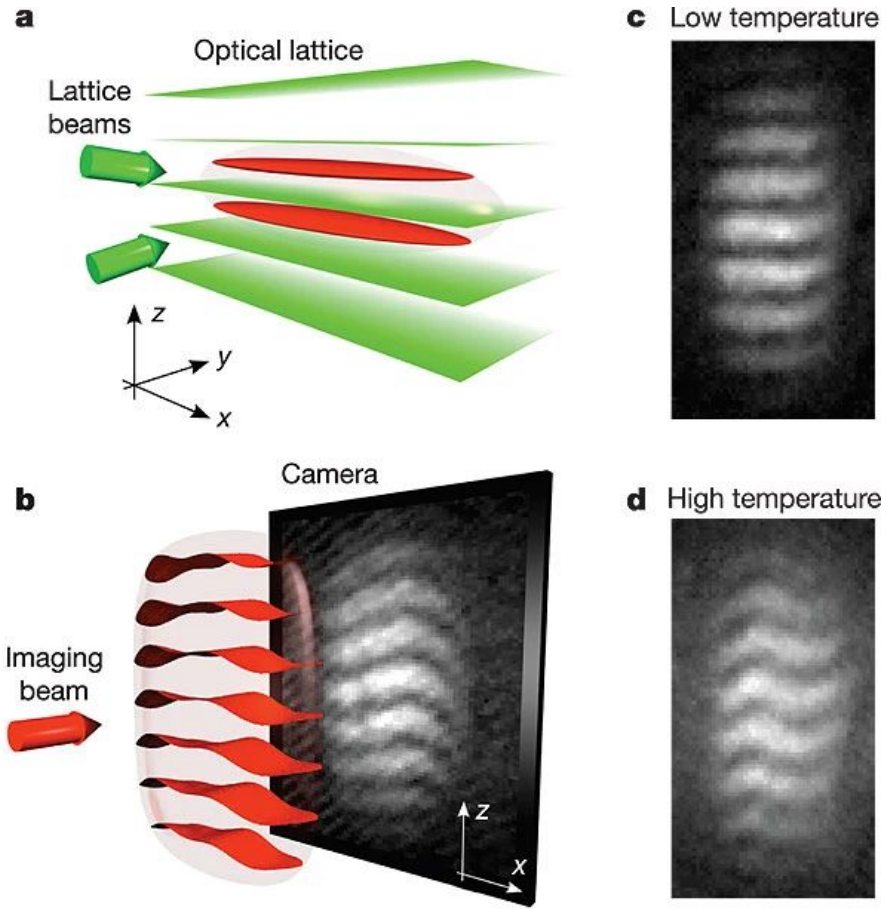
- Renormalization of SF density by RG
- Governs jump in the SF density
  - Infinite order phase transition



Giacomo Bighin et. al., *Symmetry* **2022** 14(10), 2184

Experimental detection

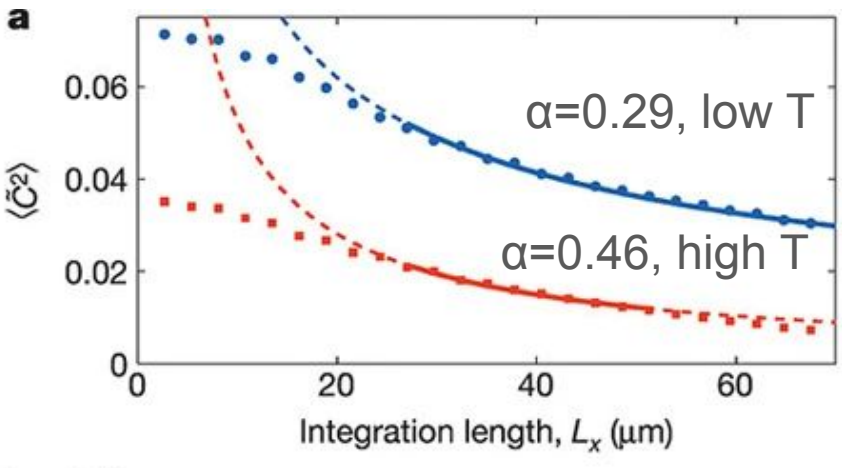


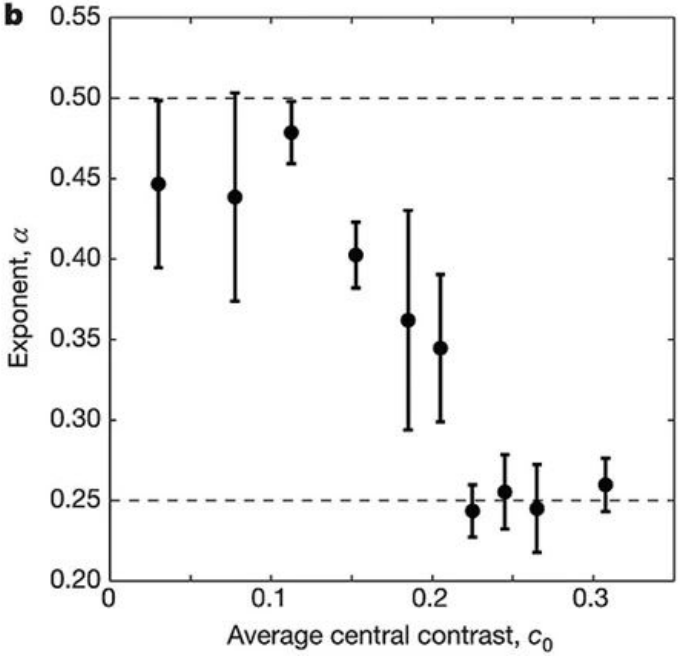


Hadzibabic, Z., Krüger, P., Cheneau, M. *et al.* Berezinskii–Kosterlitz–Thouless crossover in a trapped atomic gas. *Nature* 441, 1118–1121 (2006)

Experimental detection

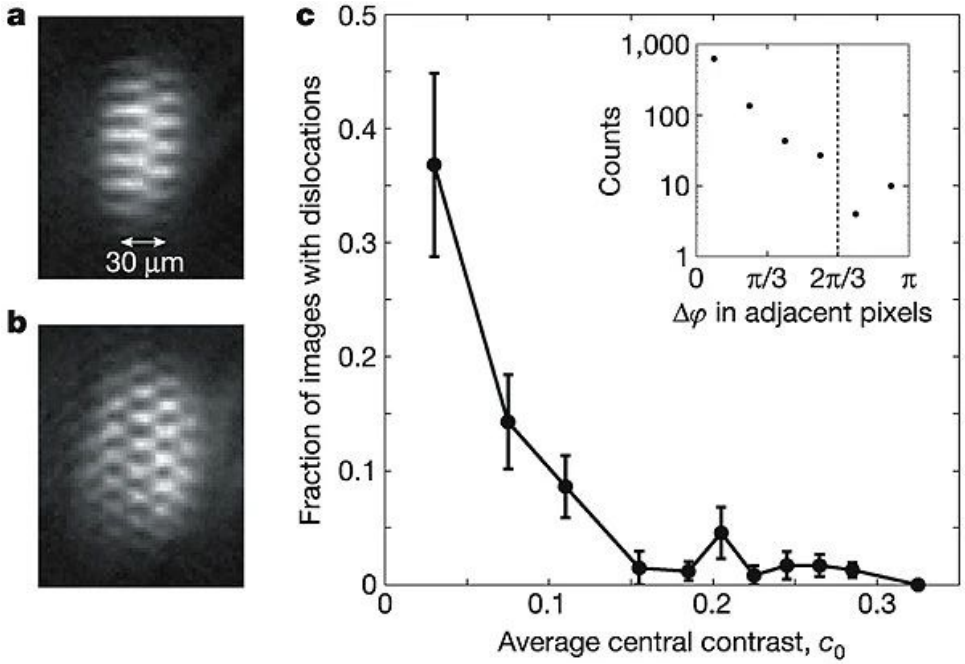
fit 1/(Lx)^2α

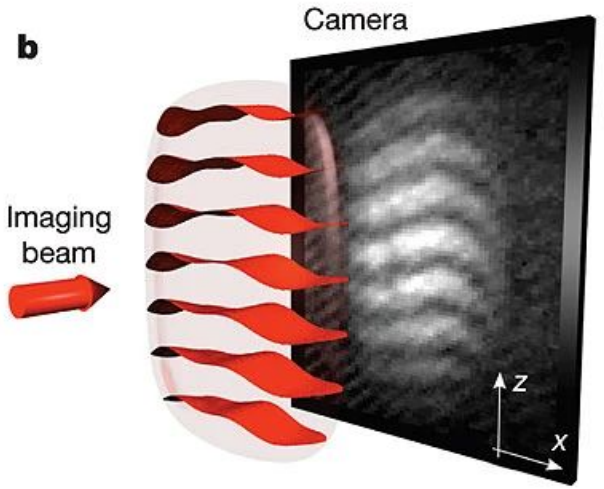




Hadzibabic, Z., Krüger, P., Cheneau, M. *et al.* Berezinskii–Kosterlitz–Thouless crossover in a trapped atomic gas. *Nature* 441, 1118–1121 (2006)

Experimental detection





a free vortex in one of the condensates should appear as a sharp dislocation in the interference pattern

Hadzibabic, Z., Krüger, P., Cheneau, M. *et al.* Berezinskii–Kosterlitz–Thouless crossover in a trapped atomic gas. *Nature* 441, 1118–1121 (2006)

THANK YOU FOR YOUR ATTENTION!