KPZ Universality Class

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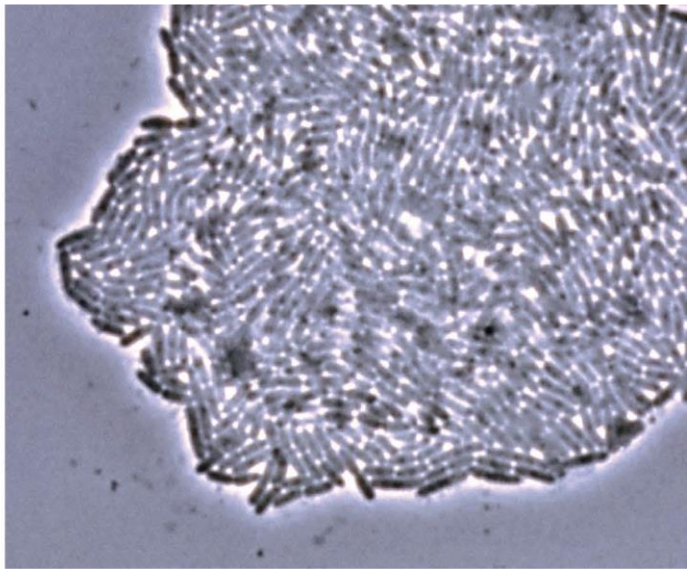
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Why KPZ?

Wildfire Fronts



Bacteria-Colony Growth



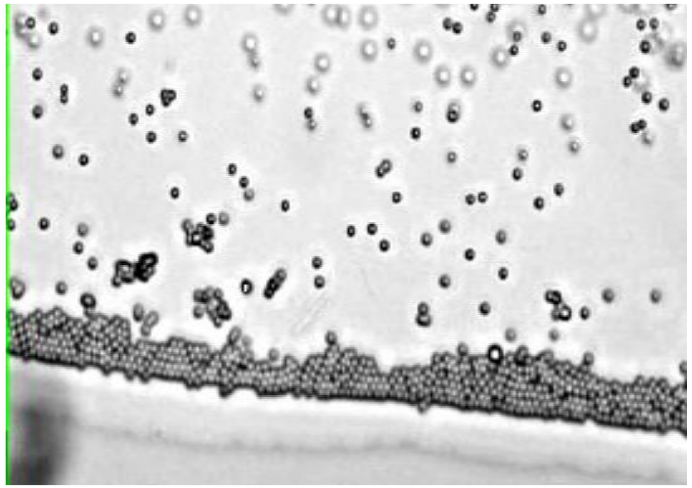
[S.Diehl, Nature **608**, 669-670 (2022)]

Fire Spreading



[Youtube]

Particle deposition



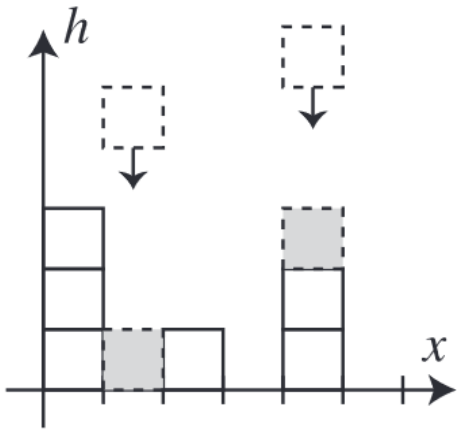
[Yunker, Nature **476**, 308-311 (2011)]

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The KPZ Equation

- Heuristic derivation:

Random depostion of blocks

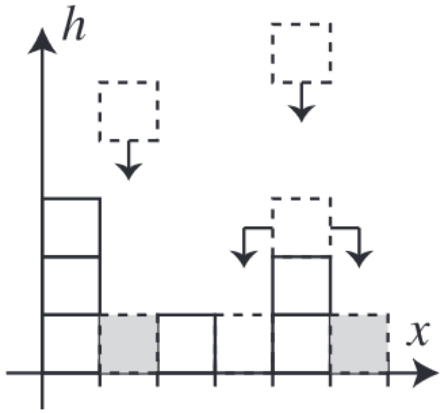


$$\rightarrow \frac{\partial}{\partial t} h(x,t) = v_0 + \eta(x,t)$$

Gaussian noise

Random depostion of blocks

with surface relaxation



$$\rightarrow \boxed{\frac{\partial}{\partial t} h(x,t) = v_0 + \nu \nabla^2 h(x,t) + \eta(x,t)}$$

Edwards-Wilkinson equation

- Linear and easy to solve exactly
- Problem: Unnatural

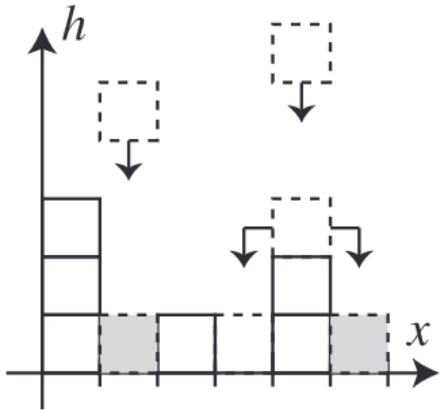
[K. A. Takeuchi, Physica A **504**, 77-105 (2018)]

The KPZ Equation

- Heuristic derivation:

Random depostion of blocks

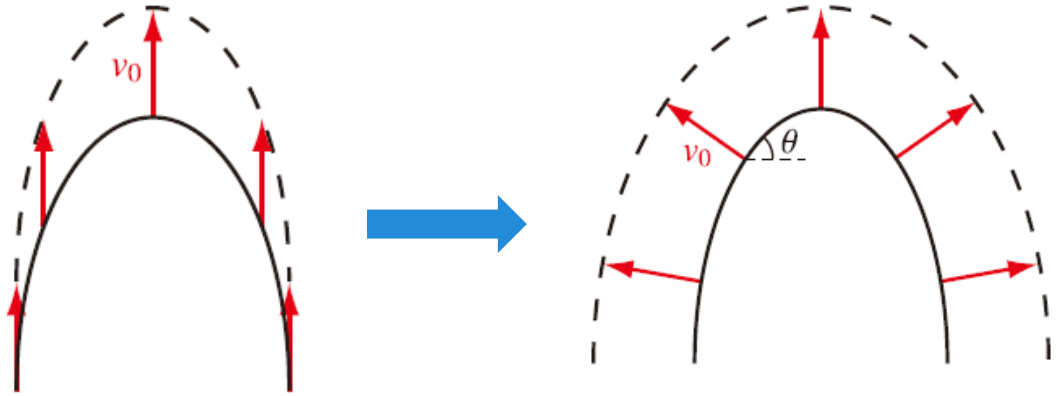
with surface relaxation



$$\frac{\partial}{\partial t}h(x,t) = v_0 + \nu\nabla^2 h(x,t) + \eta(x,t)$$

Edwards-Wilkinson equation

- Linear and easy to solve exactly
- Problem: Unnatural



$$\frac{\partial}{\partial t}h(x,t) = v_0 + \nu\nabla^2 h(x,t) + \frac{\lambda}{2}\left(\nabla h(x,t)\right)^2 + \eta(x,t)$$

Kardar-Parisi-Zhang equation

Scaling laws and Symmetries?

[K. A. Takeuchi, Physica A **504**, 77-105 (2018)]

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Solution of KPZ Equation

$$\frac{\partial}{\partial t}h(x,t) = v_0 + \nu\nabla^2 h(x,t) + \frac{\lambda}{2}\left(\nabla h(x,t)\right)^2 + \eta(x,t)$$

Cole-Hopf transformation: $Z(x,t) := e^{\frac{\lambda}{2\nu}h(x,t)}$

$$\frac{\partial}{\partial t}Z(x,t) = \nu\nabla^2 Z(x,t) + \frac{\lambda}{2\nu}Z(x,t) \times \eta(x,t)$$

Stochastic heat equation

- Ito product $G(Z,t)dB(t) \equiv \lim_{\Delta t\to 0} G(Z(t_i),t_i)[B(t_{i+1}) - B(t_i)]$
- Stratonovich product

$$G(Z,t)\circ dB(t) \equiv \lim_{\Delta t\to 0} G\left(\frac{Z(t_{i+1}) + Z(t_i)}{2}, t_i\right)[B(t_{i+1}) - B(t_i)]$$

→ Solution of KPZ-equation given by solution of stochastic heat equation

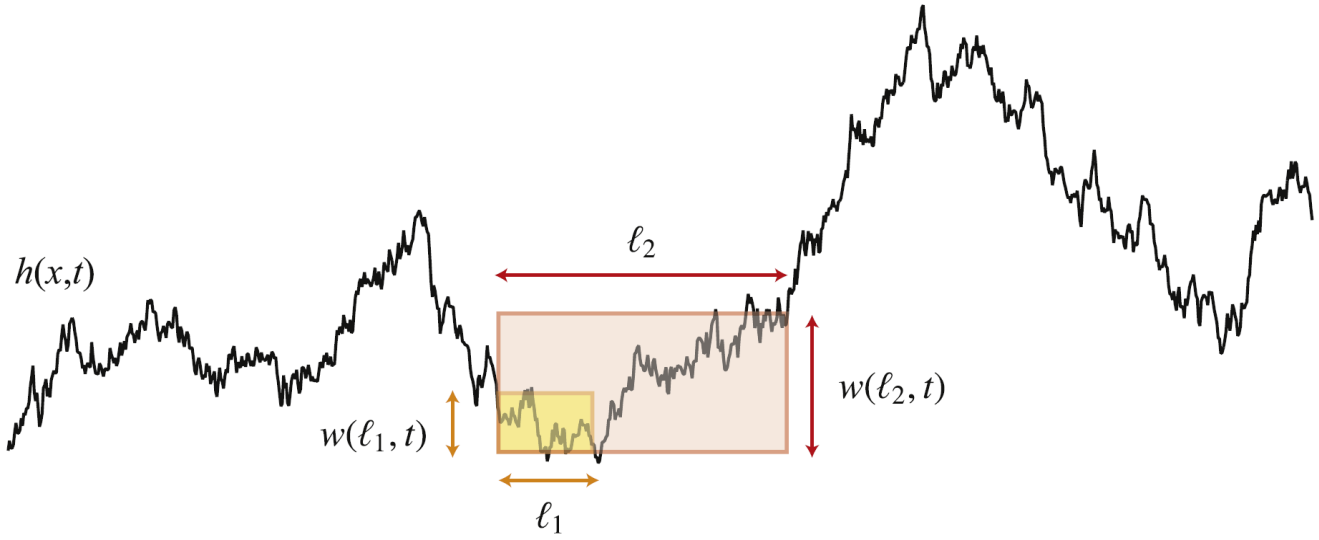
See also: [I. CORWIN, arXiv: 1106.1596v4]
[K. A. Takeuchi, Physica A **504**, 77-105 (2018)]

Scaling laws and Symmetries of KPZ Equation

$$\frac{\partial}{\partial t}h(x,t) = v_0 + \nu\nabla^2 h(x,t) + \frac{\lambda}{2}\left(\nabla h(x,t)\right)^2 + \eta(x,t)$$

- Time translation $t \to t + t_0$
- Space translation $x \to x + x_0$
- Space inversion $x \to -x$
- Height translation $h \to h + h_0$
- KPZ-equation is self-affine

$$x \to bx \qquad t \to b^z t \qquad \delta h \to b^\alpha \delta h$$



→ • In 1D $\alpha = \frac{1}{2}$ $\beta = \frac{1}{3}$ $z = \frac{\alpha}{\beta}$

→ • In 2D $\alpha = 0.3869$ $\beta = 0.2398$ $z = 1.6131$

Family-Vicsek scaling

$$w(l,t) \sim \begin{cases} l^\alpha, & l \ll \xi \\ t^\beta, & l \gg \xi \end{cases} \quad \text{with} \quad \xi \sim t^{\frac{1}{z}}$$

Strechted exponential for KPZ

[K. A. Takeuchi, Physica A **504**, 77-105 (2018)]

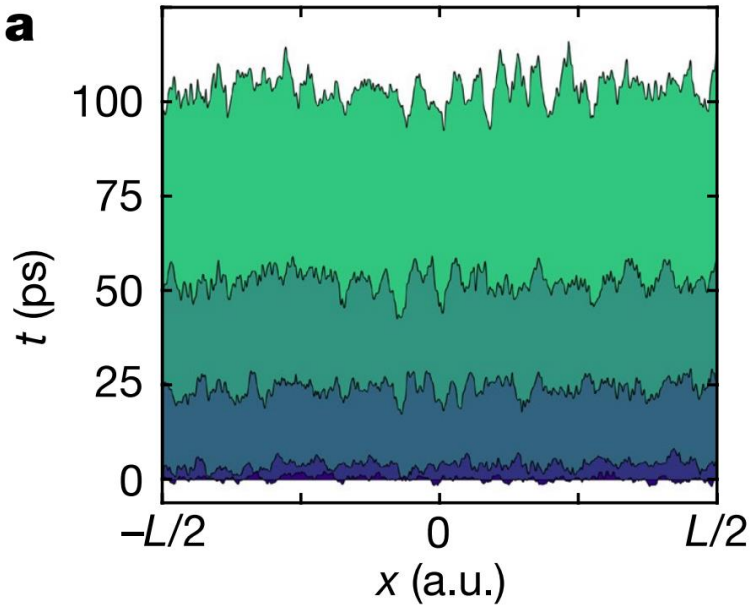
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KPZ-Scaling in Exciton-Polariton Condensate - Theory

$$\begin{cases} i\hbar\partial_t\psi = \left[E(\hat{k}) - \dfrac{i\hbar}{2}\gamma(\hat{k}) + g|\psi|^2 + 2g_R n_R + \dfrac{i\hbar}{2}R\,n_R\right]\psi + \hbar\xi \\ \partial_t n_R = P - (\gamma_R + R|\psi|^2)n_R \end{cases}$$

$$\psi(x,t) = \sqrt{\rho(x,t)}\exp\left[i(\theta(x,t) - \overline{\omega_0}t)\right]$$

$$\langle\xi(x,t)\xi^*(x',t')\rangle = 2\xi_0\delta(x-x')\delta(t-t')$$



- The condensate phase follows a KPZ equation where the condensate phase profile behaves as a classical interface.
- The condensate and exciton densities relax in a short time scale to their stationary values compared to the phase

$$\begin{cases} \partial_t\theta = \overline{\omega_0} + \dfrac{\hbar}{2m}\Re\{\mathcal{D}[\rho,\theta]\} + \dfrac{\gamma_2}{2}\Im\{\mathcal{D}[\rho,\theta]\} - \dfrac{g}{\hbar}\rho - 2\dfrac{g_R}{\hbar}n_R + \Re\{\bar{\xi}\} \\ \dfrac{1}{2\rho}\partial_t\rho = -\dfrac{\hbar}{2m}\Im\{\mathcal{D}[\rho,\theta]\} + \dfrac{\gamma_2}{2}\Re\{\mathcal{D}[\rho,\theta]\} + \dfrac{1}{2}\Big(R\,n_R - \gamma_0\Big) + \Im\{\bar{\xi}\} \\ \partial_t n_R = P - (\gamma_R + R\,\rho)\,n_R \end{cases}$$

$$\begin{cases} \partial_t\theta = -\dfrac{\hbar}{2m}(\nabla\theta)^2 + \dfrac{\gamma_2}{2}\nabla^2\theta - \dfrac{g}{\hbar}\delta\rho - 2\dfrac{g_R}{\hbar}\delta n_R + \Re\{\bar{\xi}\} \\ \delta n_R = \dfrac{2}{R}\left[\dfrac{\hbar}{2m}\nabla^2\theta + \dfrac{\gamma_2}{2}(\nabla\theta)^2 - \Im\{\bar{\xi}\}\right] \\ \delta\rho = -p\dfrac{\gamma_R}{\gamma_0}\delta n_R\,. \end{cases}$$

[Q. Fontaine et alii, Nature **608** (2022)]

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KPZ-Scaling in Exciton-Polariton Condensate - Theory

In the vicinity of $k = 0$ $\gamma(k \simeq 0) \simeq \gamma_0 + \gamma_2 k^2$

$$\partial_t \theta = \left[\frac{\gamma_2}{2} - u\frac{g_R}{\hbar R}\frac{\hbar}{m}\right]\nabla^2\theta - \left[\frac{\hbar}{2m} + u\frac{g_R}{\hbar R}\gamma_2\right](\nabla\theta)^2 + \eta$$
$$\equiv \nu\nabla^2\theta + \frac{\lambda}{2}(\nabla\theta)^2 + \eta$$

$$\langle \eta(x,t)\eta(x',t')\rangle = \frac{\xi_0}{\rho_0}\left[1 + 4\left(u\frac{g_R}{\hbar R}\right)^2\right]\delta(x-x')\delta(t-t')$$
$$\equiv 2D\delta(x-x')\delta(t-t')\,,$$

$$u = 2 - p\frac{g}{g_R}\frac{\gamma_R}{\gamma_0}\,.$$

$$\nu = \frac{\gamma_2}{2} - u\frac{g_R}{\hbar R}\frac{\hbar}{m}\,, \quad \lambda = -2\left[\frac{\hbar}{2m} + u\frac{g_R}{\hbar R}\gamma_2\right]\,, \quad D = \frac{\xi_0}{2\rho_0}\left[1 + 4\left(u\frac{g_R}{\hbar R}\right)^2\right] = \frac{R\,n_{R,0}}{4\rho_0}\left[1 + 4\left(u\frac{g_R}{\hbar R}\right)^2\right]$$

[Q. Fontaine et alii, Nature **608** (2022)]

KPZ-Scaling in Exciton-Polariton Condensate - Theory

$$g^{(1)}(\Delta x, \Delta t) = \frac{\langle \psi^*(x, t_0)\psi(-x, t_0 + \Delta t)\rangle}{\sqrt{\langle \rho(x, t_0)\rangle}\sqrt{\langle \rho(-x, t_0 + \Delta t)\rangle}}$$
$$= \frac{\left\langle \sqrt{\rho(x, t_0)\rho(-x, t_0 + \Delta t)}e^{i\Delta\theta(\Delta x, \Delta t)}\right\rangle}{\sqrt{\langle \rho(x, t_0)\rangle}\sqrt{\langle \rho(-x, t_0 + \Delta t)\rangle}}.$$

$$g^{(1)}(\Delta x, \Delta t) = \frac{\left\langle \sqrt{\rho(x, t_0)\rho(-x, t_0 + \Delta t)}\right\rangle}{\sqrt{\langle \rho(x, t_0)\rangle}\sqrt{\langle \rho(-x, t_0 + \Delta t)\rangle}} \left\langle e^{i\Delta\theta(\Delta x, \Delta t)}\right\rangle$$

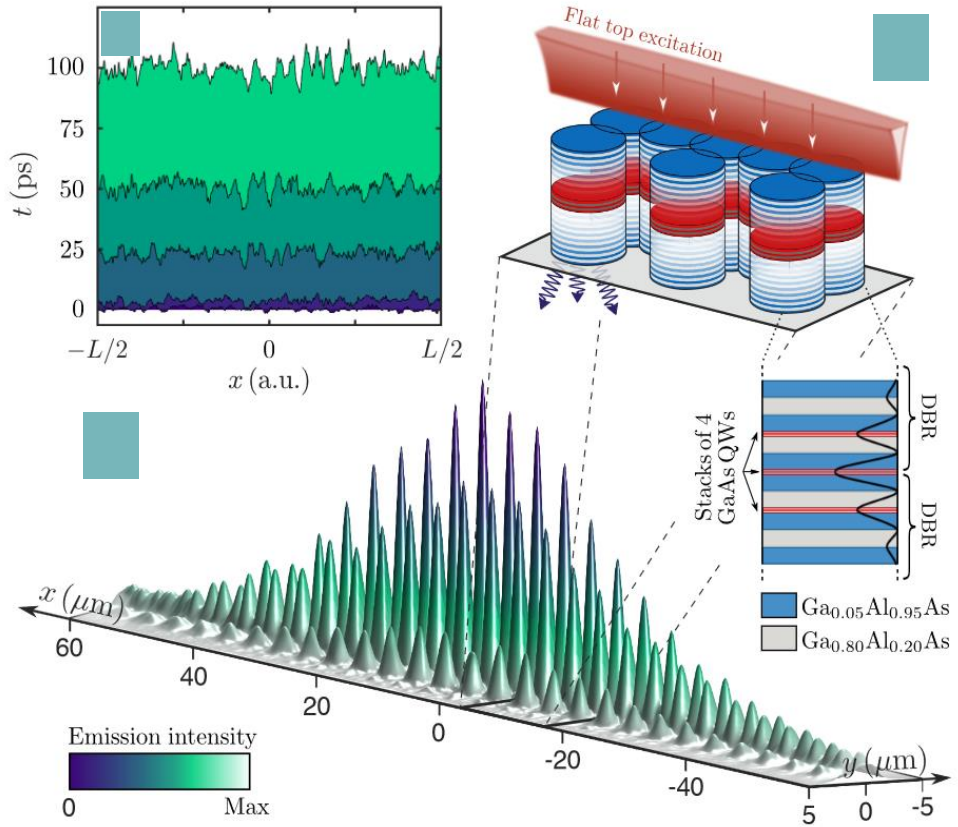
$$g^{(1)}(\Delta x, \Delta t) = \left\langle \exp\left[i\Delta\theta(\Delta x, \Delta t)\right]\right\rangle.$$

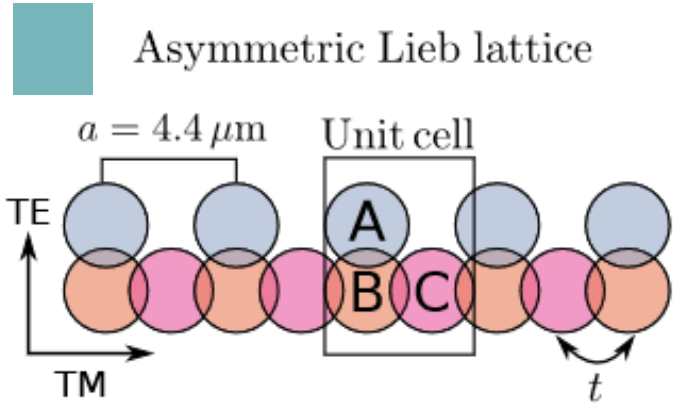
$$|g^{(1)}(\Delta x, 0)| \propto \exp[-(\Delta x/\lambda)^{2\chi}/2] \qquad |g^{(1)}(0, \Delta t)| \propto \exp[-(\Delta t/\tau)^{2\beta}/2]$$

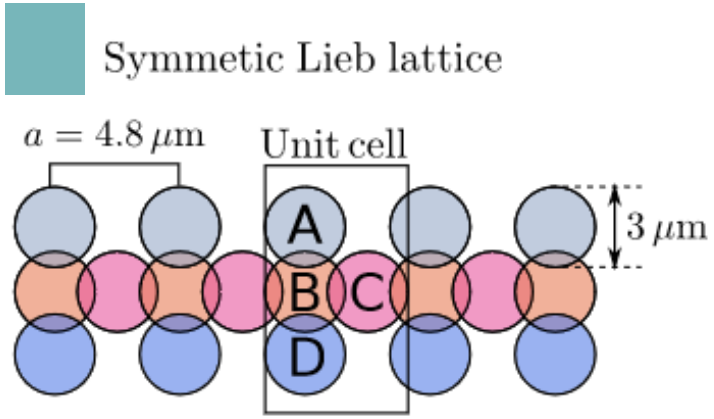
[Q. Fontaine et alii, Nature **608** (2022)]

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KPZ-Scaling in Exciton-Polariton Condensate - Experiment



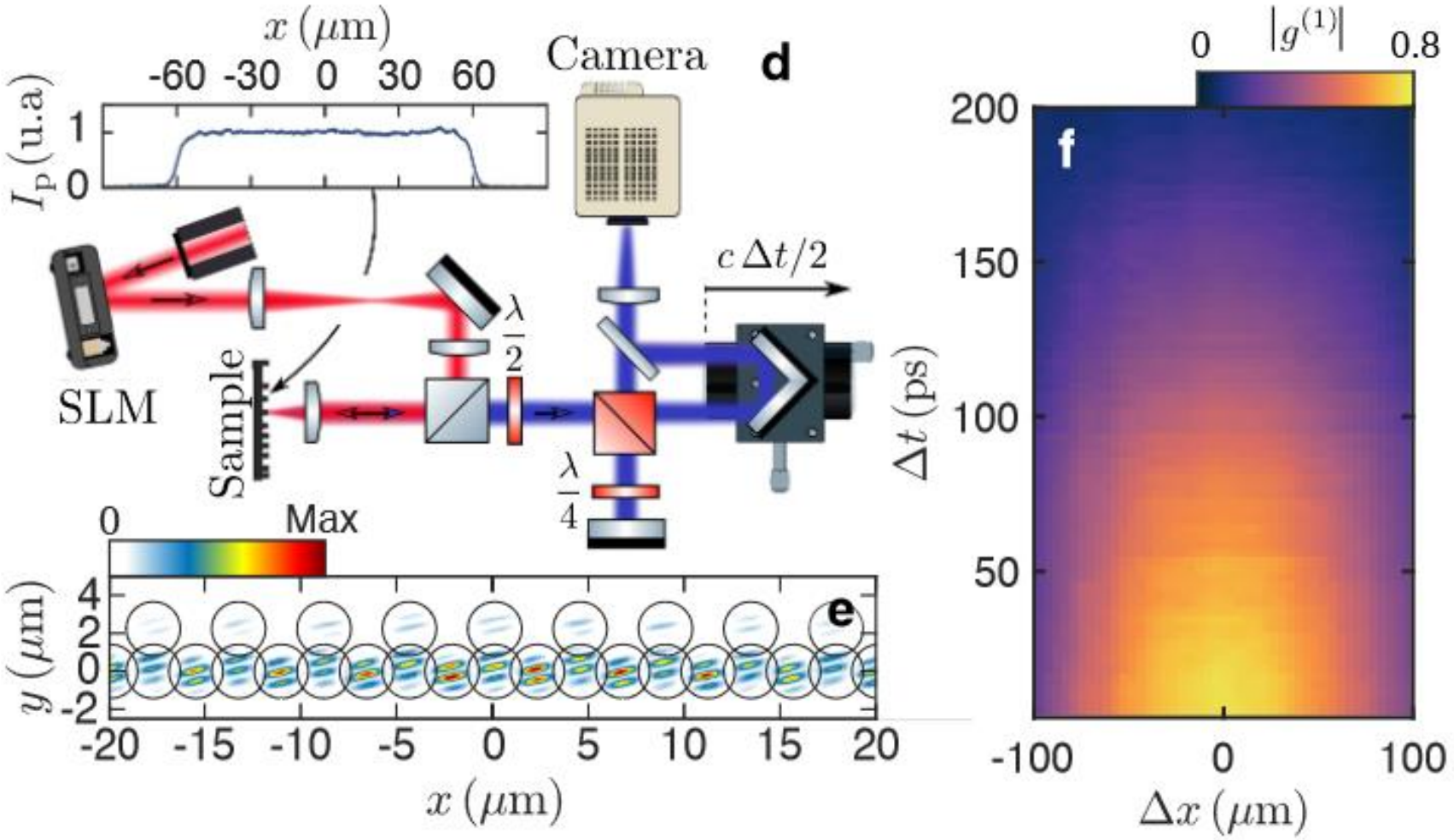




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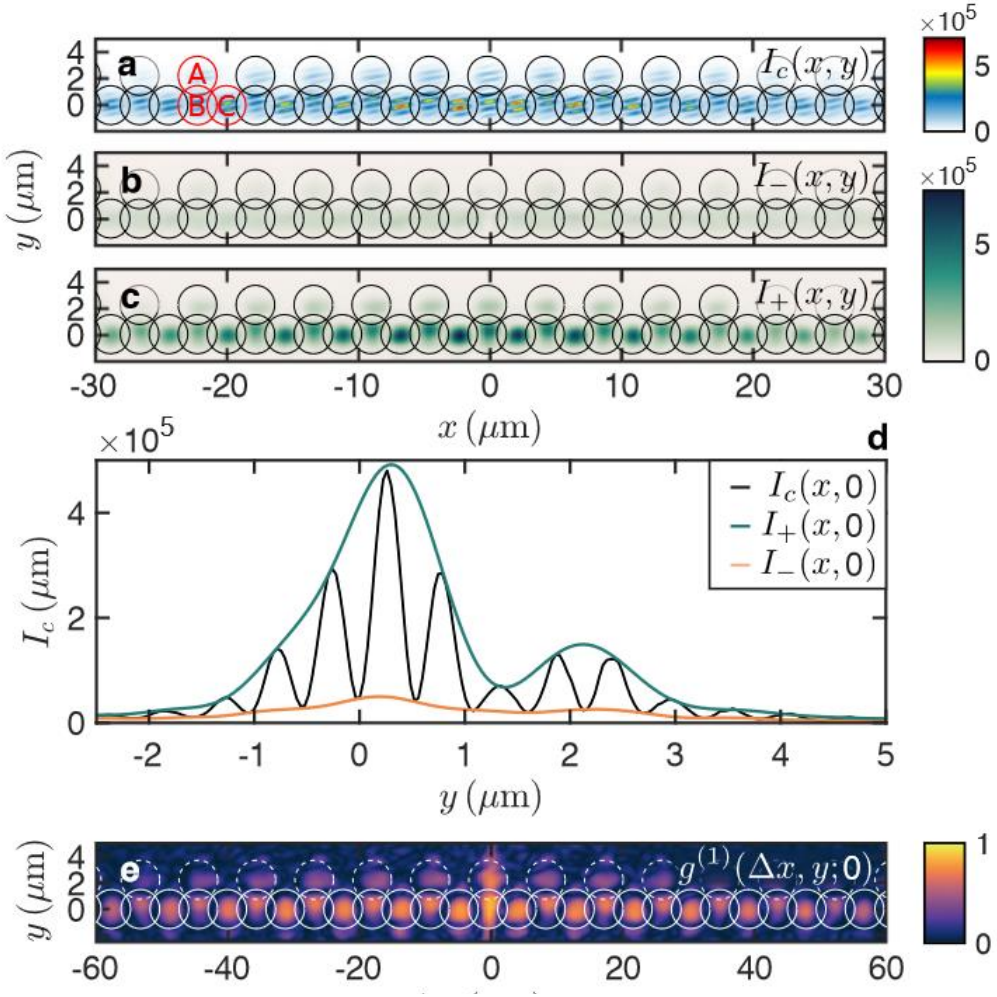
KPZ-Scaling in Exciton-Polariton Condensate - Experiment

$$g^{(1)}(\Delta x, \Delta t) = \left\langle \exp\left[i\Delta\theta(\Delta x, \Delta t)\right] \right\rangle \longrightarrow -2\log(|g^{(1)}(\Delta x, \Delta t)|) \simeq \mathrm{Var}\left[\Delta\theta(\Delta x, \Delta t)\right]$$



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KPZ-Scaling in Exciton-Polariton Condensate - Experiment

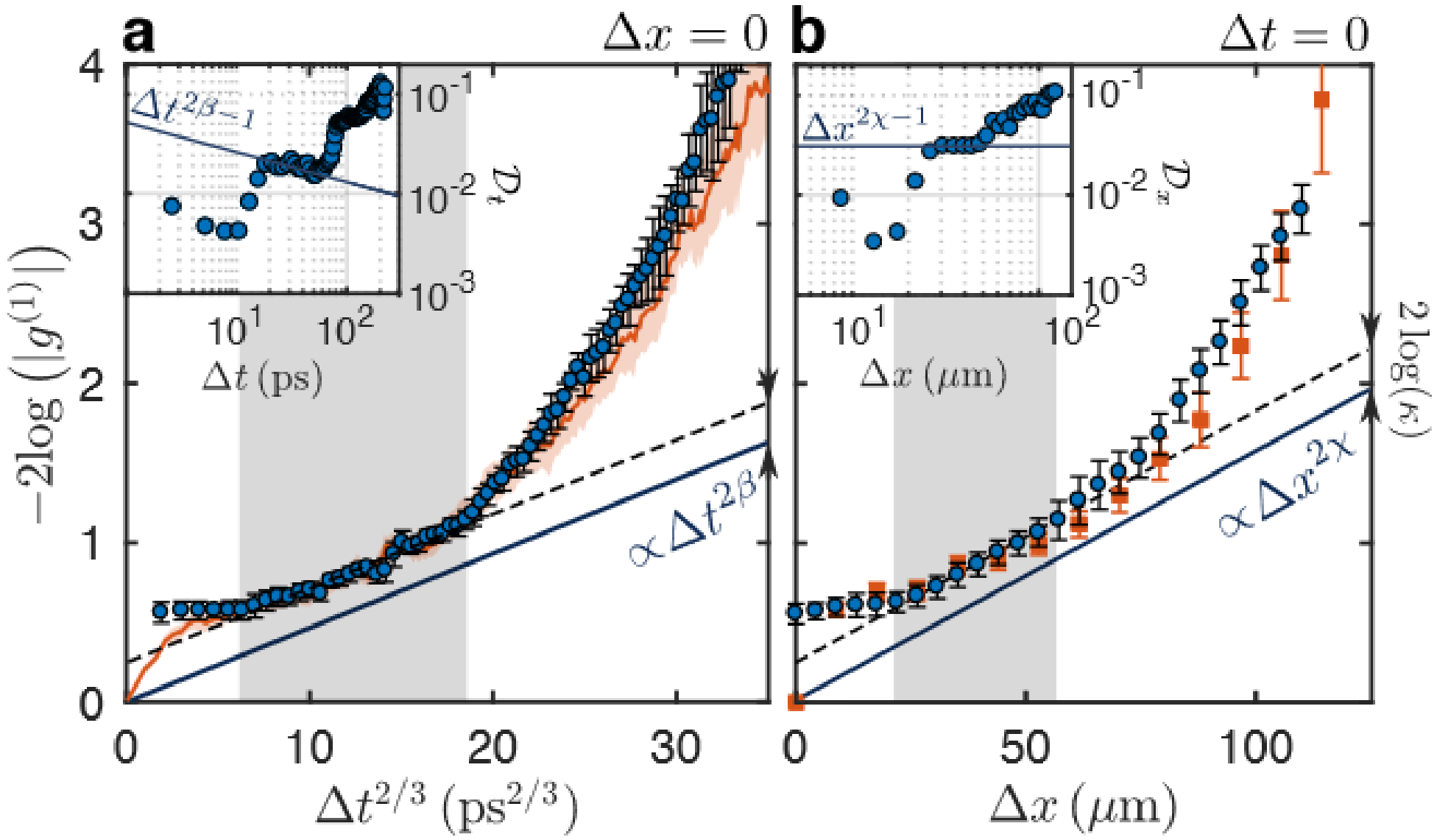


$$V = (I_+ - I_-)/(I_+ + \mathrm{I}_-)$$

$$V(\Delta x, \Delta t) = \frac{2\sqrt{I(x)I(-x)}}{I(x)+I(-x)} |g^{(1)}(\Delta x, \Delta t)| = K(\Delta x) |g^{(1)}(\Delta x, \Delta t)|$$

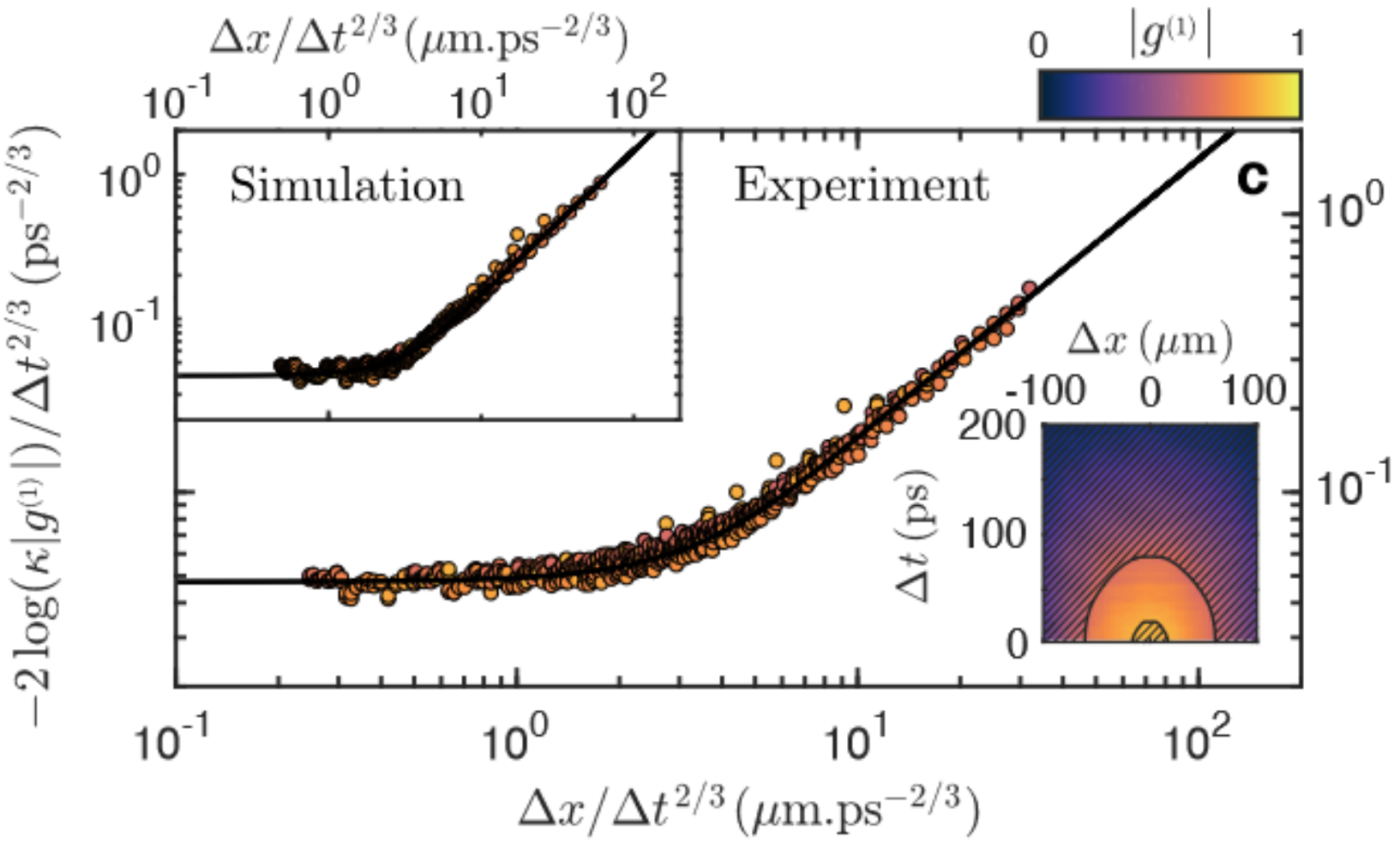
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KPZ-Scaling in Exciton-Polariton Condensate - Experiment



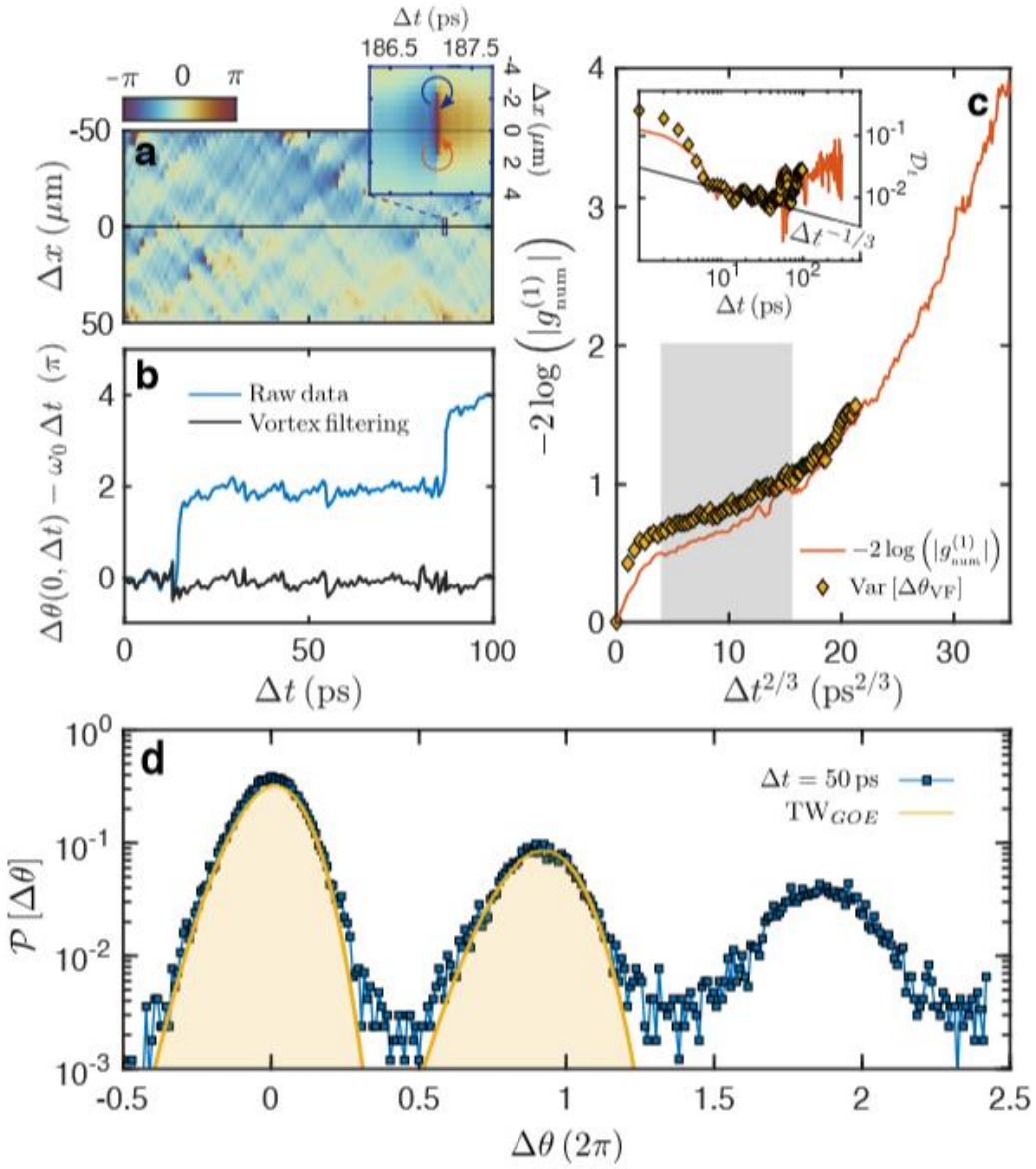
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KPZ-Scaling in Exciton-Polariton Condensate - Experiment



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KPZ-Scaling in Exciton-Polariton Condensate - Experiment



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