# From BCS to Bose Superfluidity in Two-dimensional Fermi Gases:

## Lecture I:
## BKT Phase Diagram for BCS-Bose in 2D

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Georgia Institute of Technology

**Kaiserslautern, Germany: August 21th, 2024**

# Outline

- Introduction: from BCS to Bose superfluidity.
- Creating ultra-cold fermions in the flatland.
- BCS to Bose superfluidity in two-dimensions.
- Conclusions.

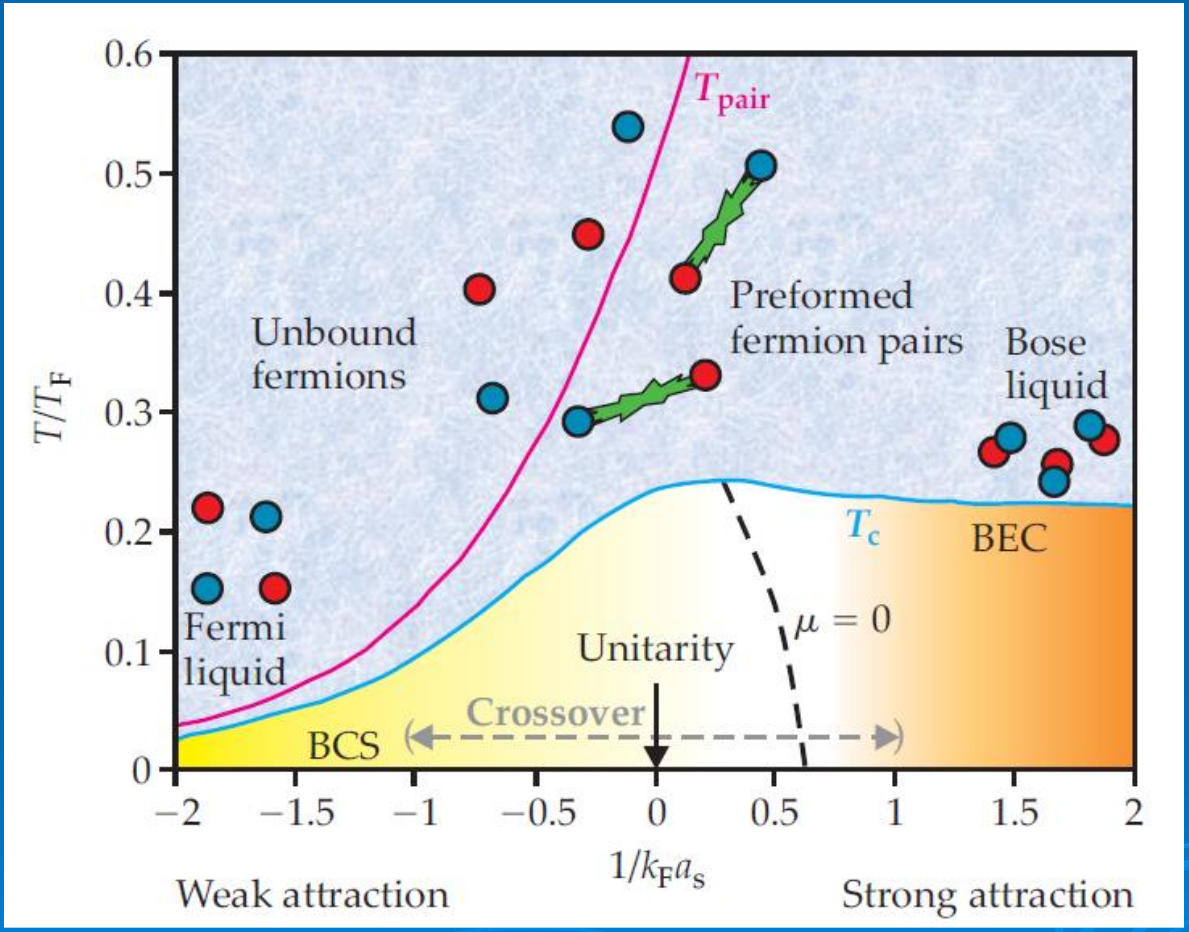
# Conclusions in Words

- The evolution from BCS to Bose superfluidity in 3D s-wave systems is analytic, i.e., a crossover.

- The evolution from BCS to Bose superfluidity in 2D s-wave systems is non-analytic, i.e., there is a phase transition, where the BKT state at low temperature may form a vortex-antivortex lattice.

# Conclusions in Pictures

SdM Physics Today 2008

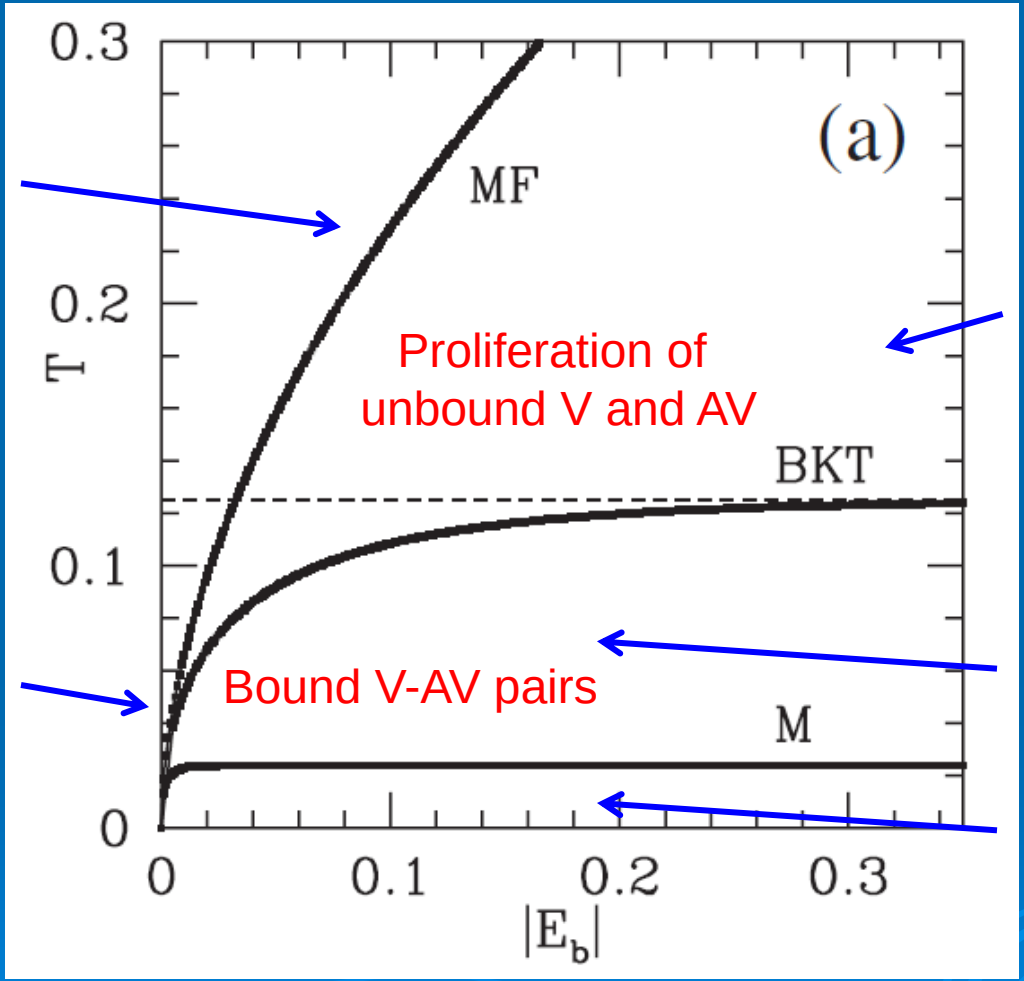



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BCS-Bose Superfluidity in 3D

# Conclusions in Pictures

Pairing Temperature




Bose Liquid

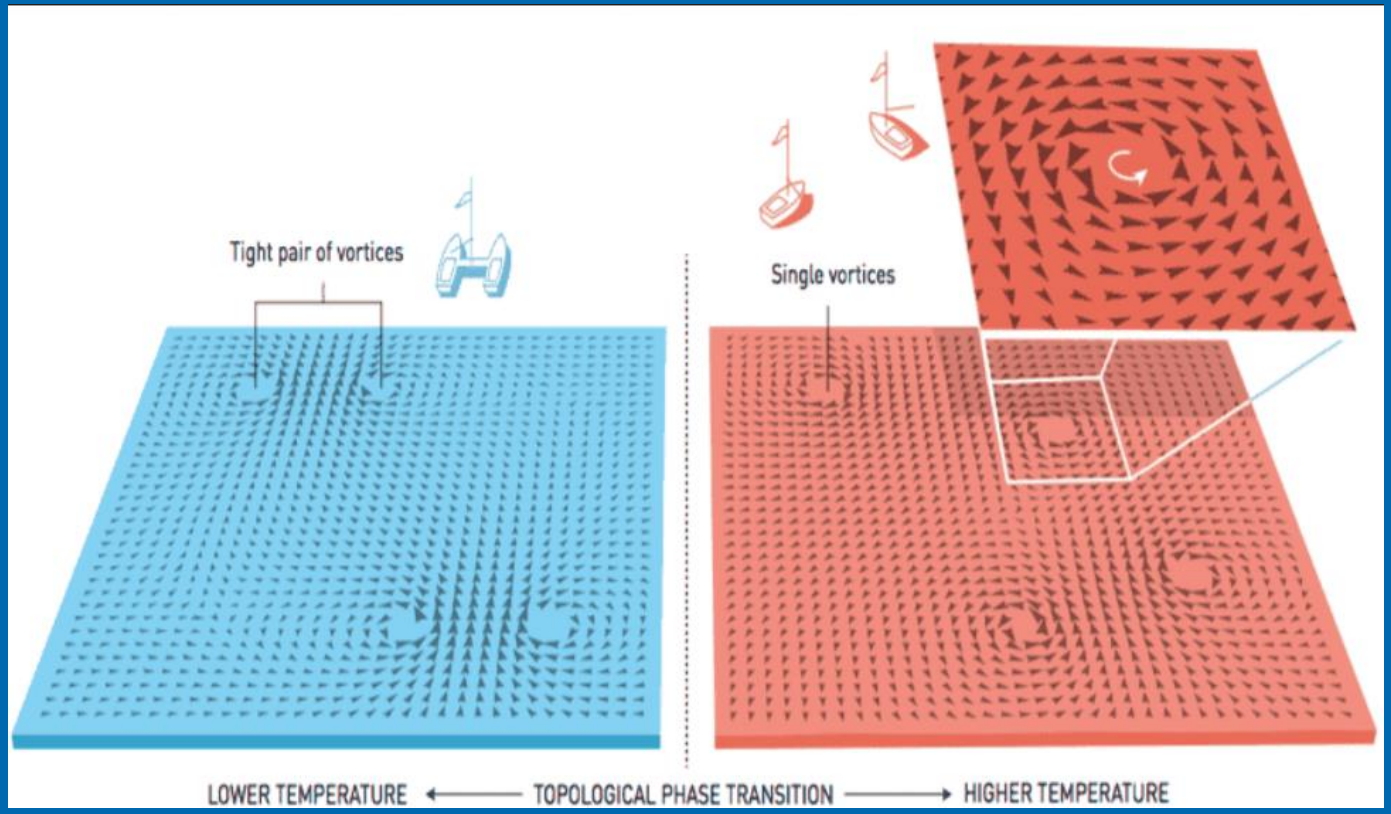
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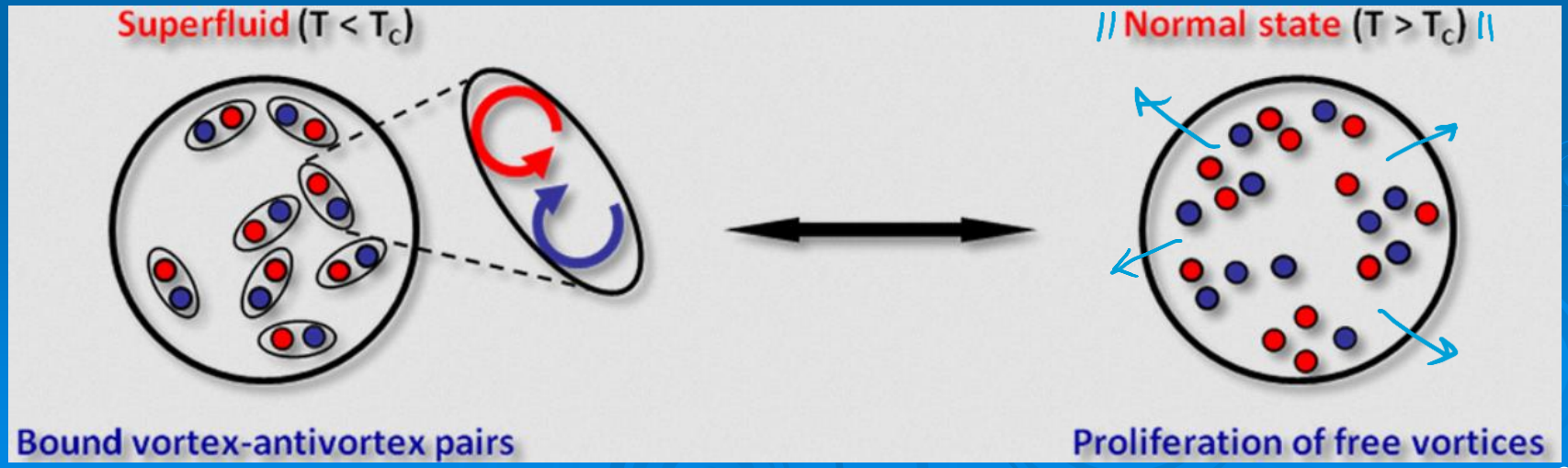
Fermi Liquid

V-AV Liquid

V-AV Lattice

BCS-Bose Superfluidity in 2D


Tight pair of vortices
Single vortices
LOWER TEMPERATURE
TOPOLOGICAL PHASE TRANSITION
HIGHER TEMPERATURE


Superfluid (T < Tc)
Normal state (T > Tc)
Bound vortex-antivortex pairs
Proliferation of free vortices

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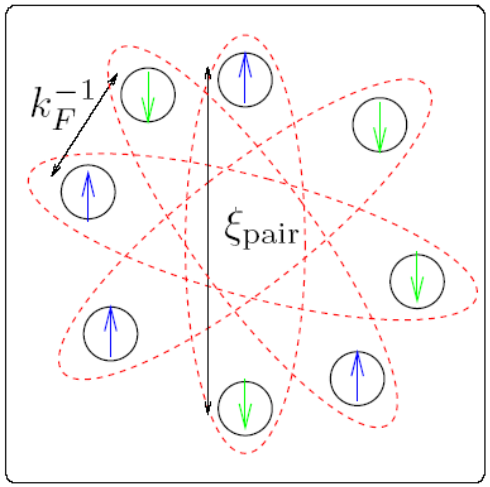
# What does the evolution from BCS to BEC superfluidity means?

**BCS regime:**

- Large "atomic pairs" ($\xi_{\rm pair} \gg k_F^{-1}$)
- Higher $B$-field side of the resonance
- Weak effective coupling ($\lambda < \lambda_c$)
- Negative scattering length ($a_\ell < 0$)

**BEC regime:**

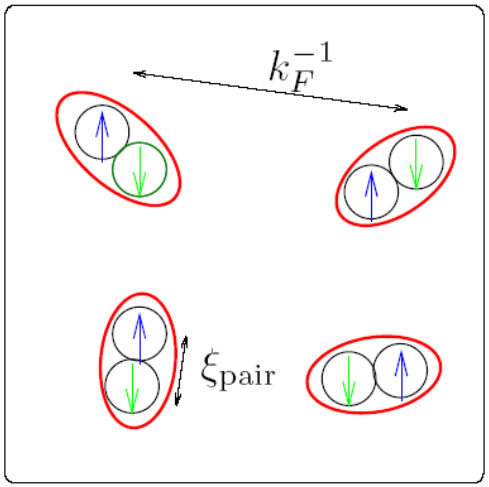
- Small "Bosonic molecules" ($\xi_{\rm pair} \ll k_F^{-1}$)
- Lower $B$-field side of the resonance
- Strong effective coupling ($\lambda > \lambda_c$)
- Positive scattering length ($a_\ell > 0$)




**Diluteness Condition**

$(k_0/k_F)^3 >> 1$

$(L_F/R_0)^3 >> 1$



# Simplest Hamiltonian: Contact Interactions

$$H = \int d\mathbf{x} \sum_{\sigma} \Psi_{\sigma}^{*}(\mathbf{x}) \left[ \frac{-\nabla^2}{2m} \right] \Psi_{\sigma}(\mathbf{x}) - g \int d\mathbf{x}\, \Psi_{\uparrow}^{*}(\mathbf{x}) \Psi_{\downarrow}^{*}(\mathbf{x}) \Psi_{\downarrow}(\mathbf{x}) \Psi_{\uparrow}(\mathbf{x})$$

Diluteness condition

$$\left(k_0 / k_F\right)^3 >> 1$$

$$\left(L_{\mathrm{F}} / R_0\right)^3 >> 1$$

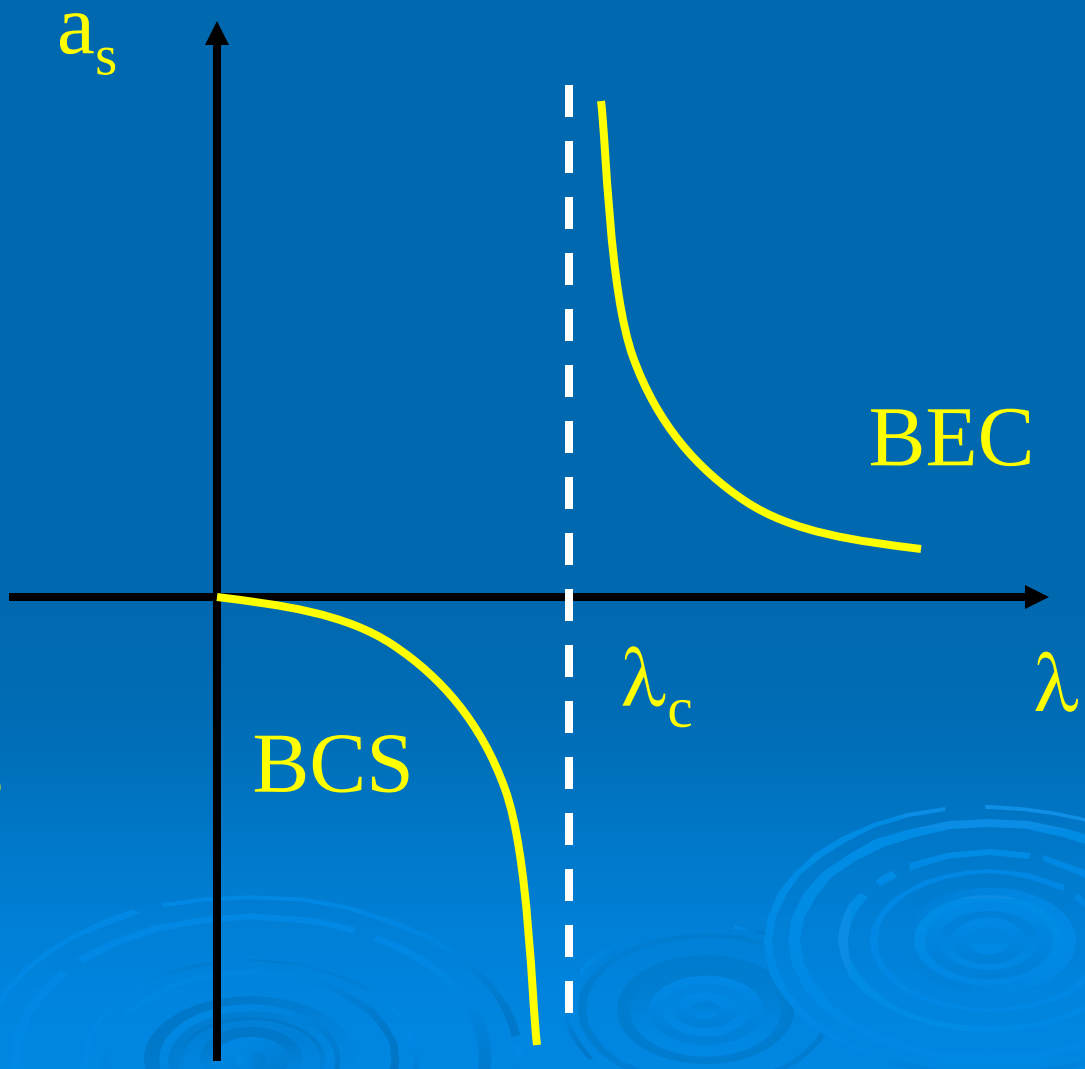
Lippman - Schwinger (T - Matrix)

$$\frac{V}{g} = -\frac{mV}{4\pi a_s} + \sum_{\mathbf{k}} \frac{1}{2\varepsilon_{\mathbf{k}}}$$

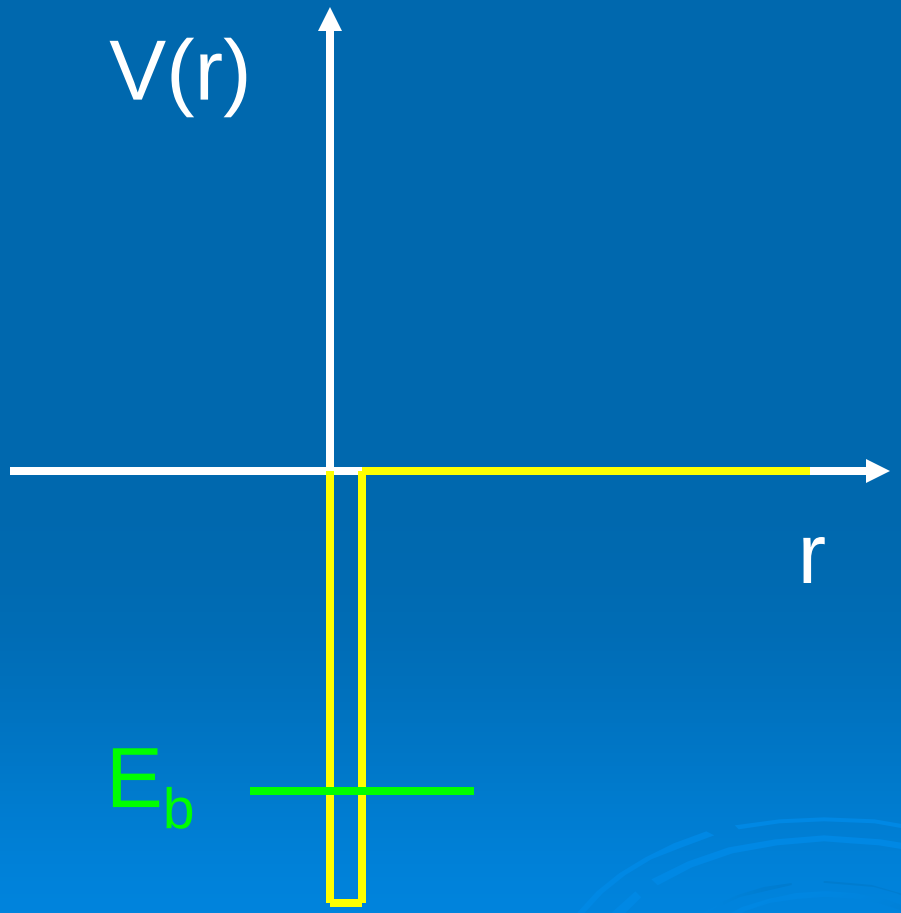
# Scattering Length

$\lambda = g/V$

Dimensions of energy.



# Two-particle bound state beyond $g > g_c$ where $a_s = \infty$.

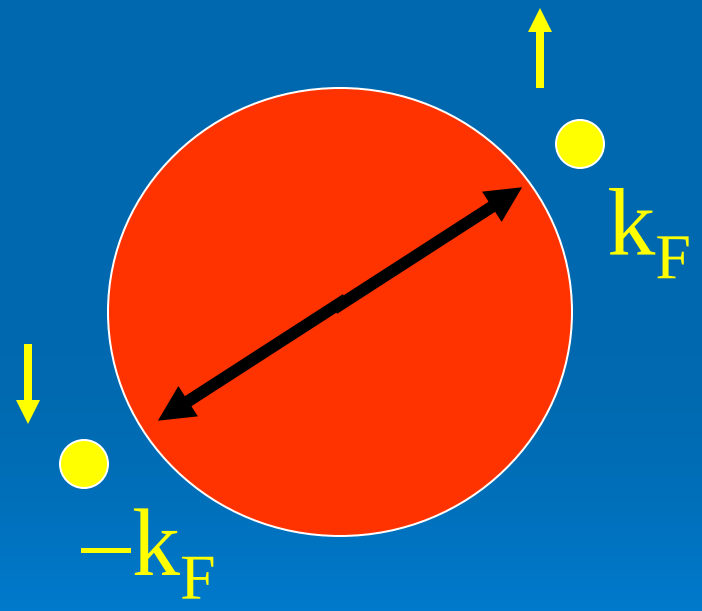


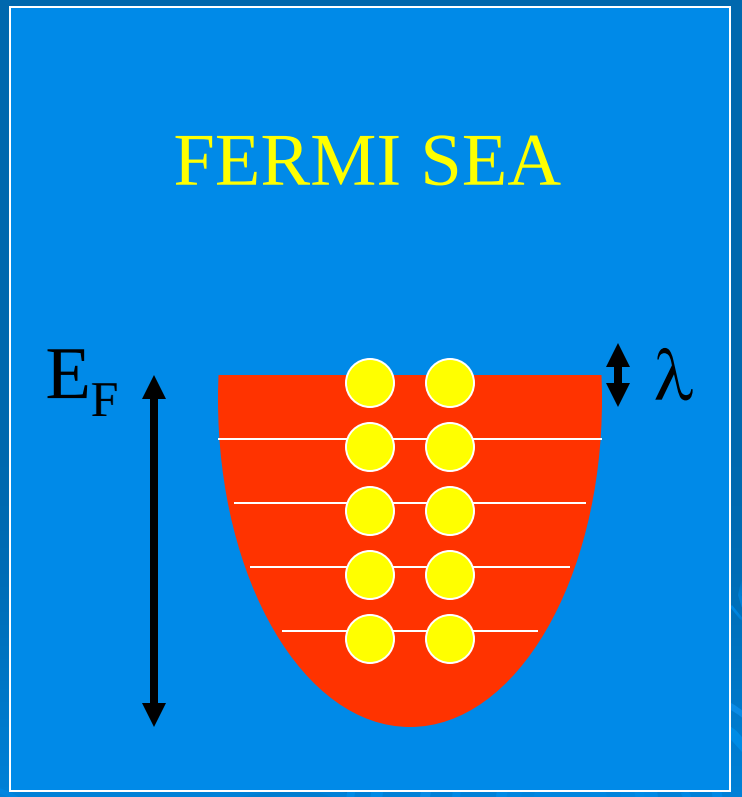

$$\frac{V}{g} = -\frac{mV}{4\pi a_s} + \sum_{\mathbf{k}} \frac{1}{2\varepsilon_{\mathbf{k}}}$$

$$E_b = -\frac{1}{ma_s^2}$$

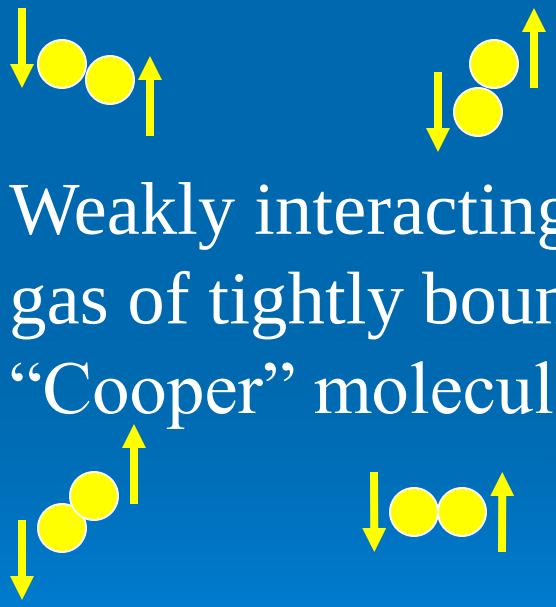
# BCS Pairing ($\lambda << E_F$ or $k_F a_s \rightarrow 0^-$)

$\mu = E_F > 0$

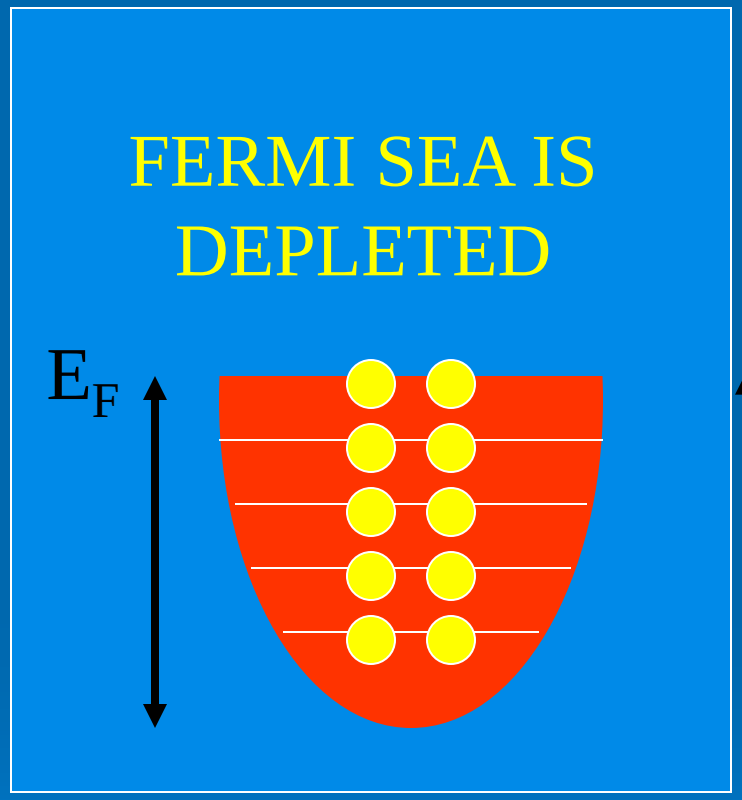


# BEC Pairing ($\lambda >> E_F$ or $k_F a_s \rightarrow 0^+$)

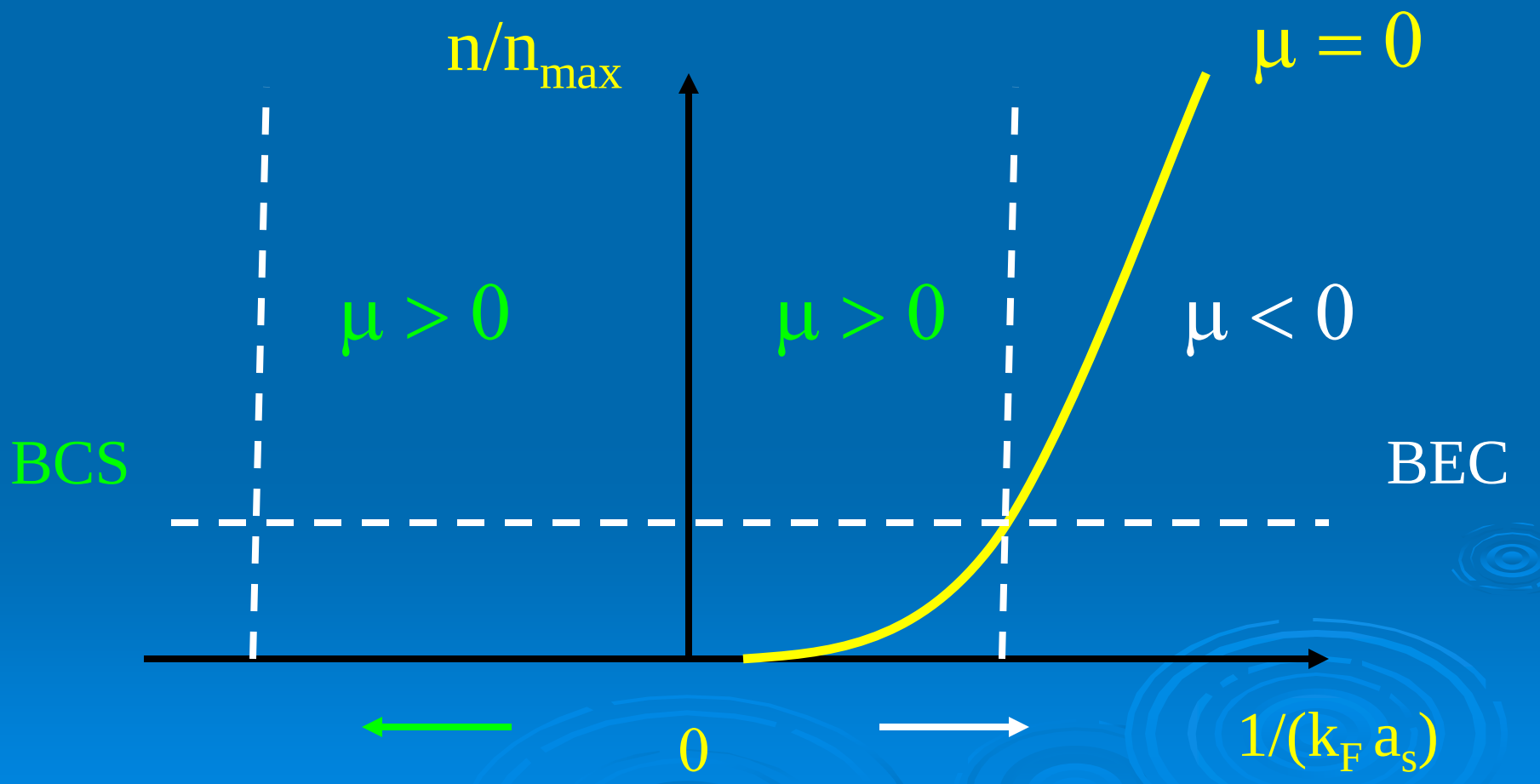


Weakly interacting gas of tightly bound "Cooper" molecules

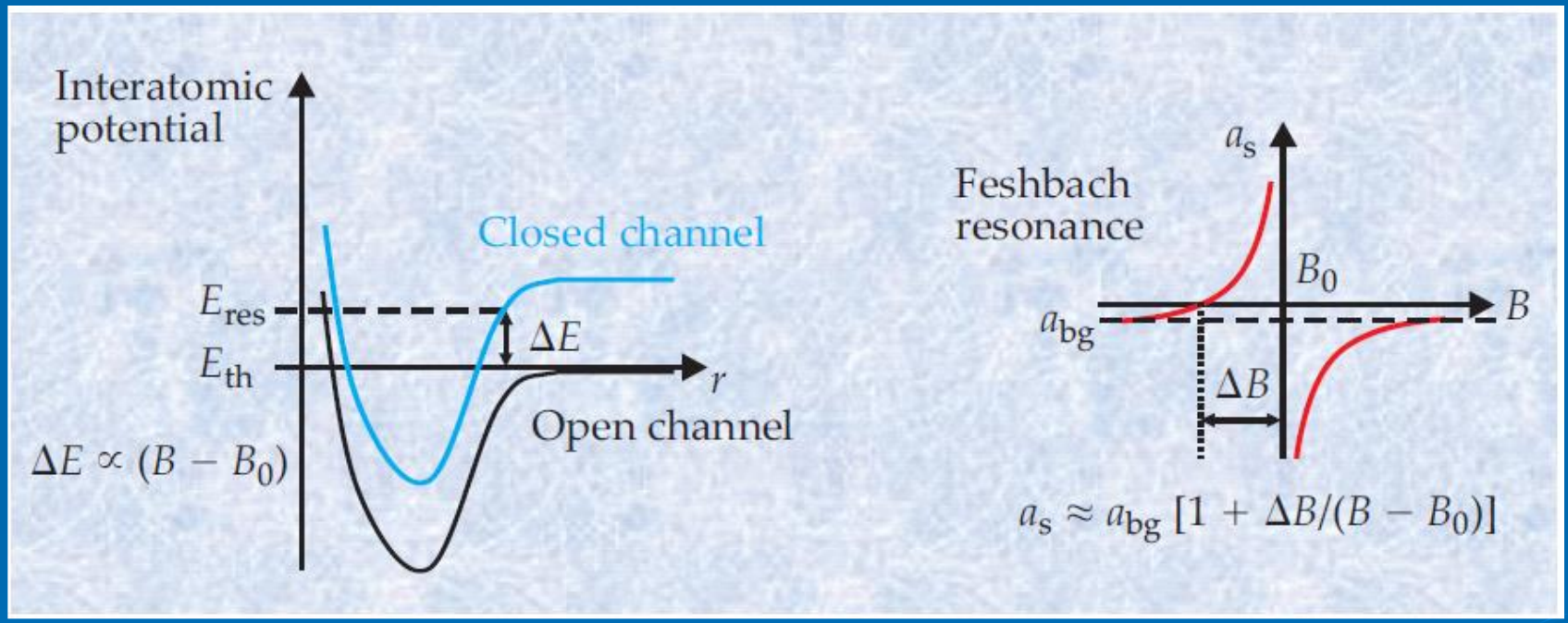



$\mu = -E_b/2 < 0$

# Schematic T = 0 Phase Diagram: Density versus Interaction



# Feshbach Resonances

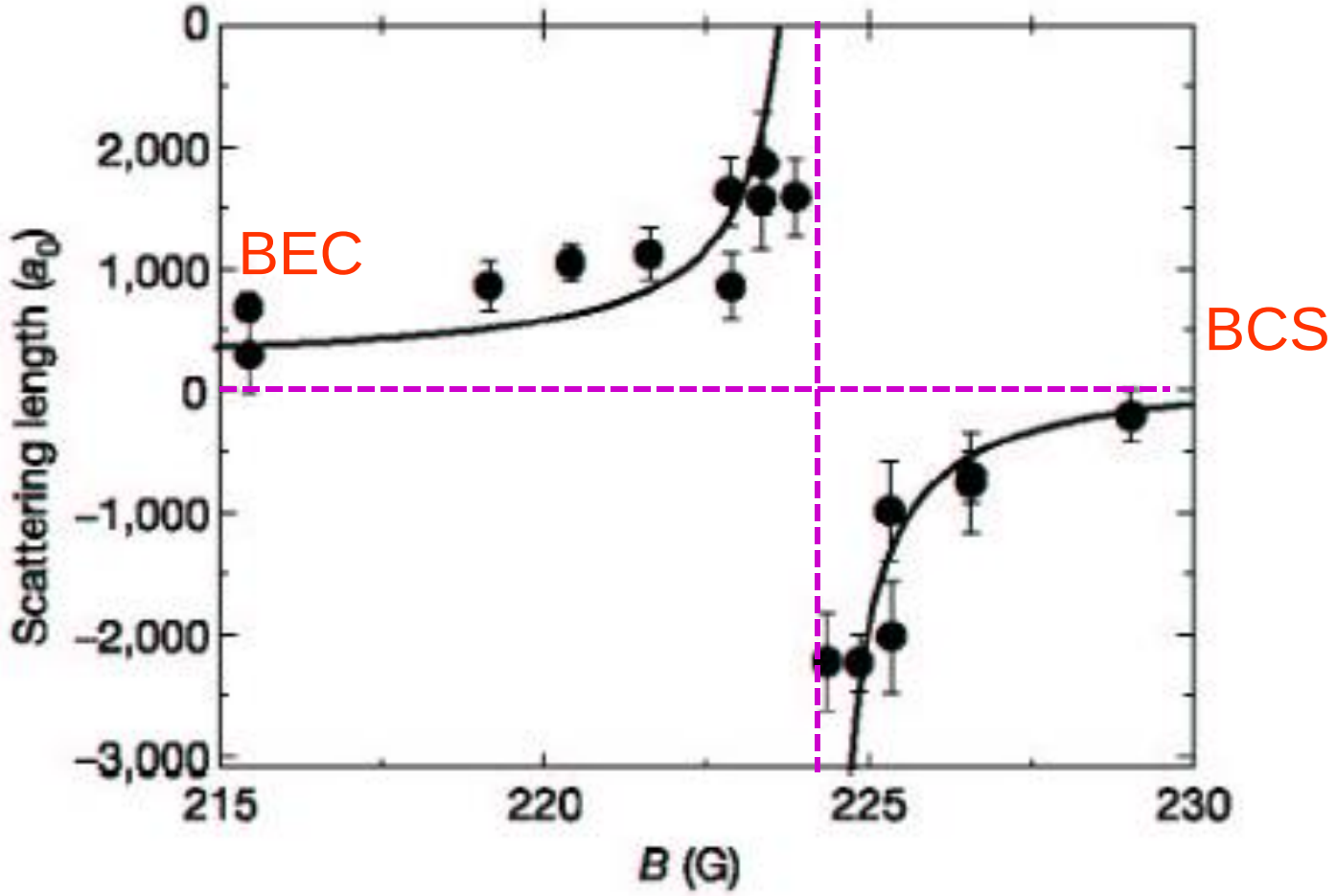



Contact interaction

$$g \rightarrow a_s \rightarrow a_s(B)$$

B-dependent scattering length

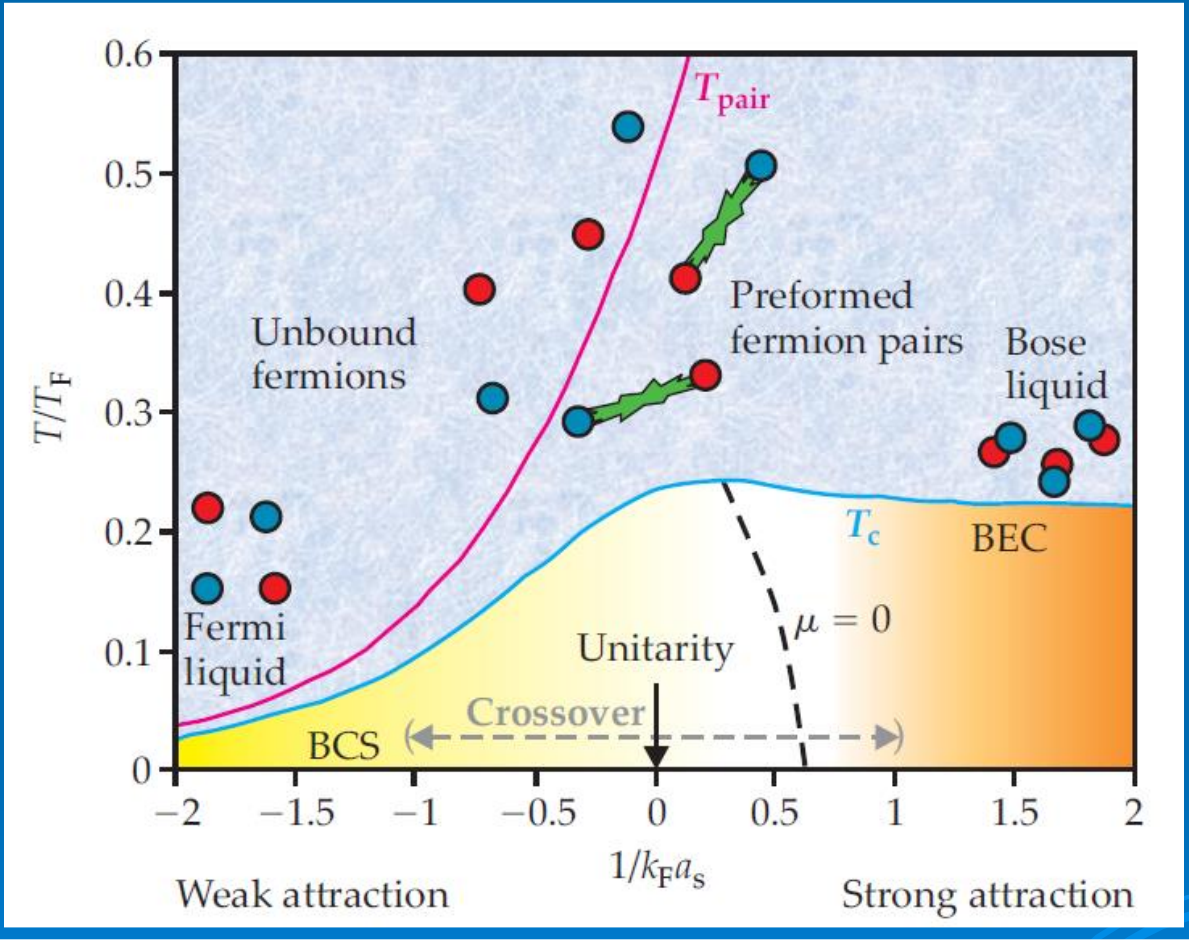
# S-wave scattering length ($^{40}$K)




Regal et. al., PRL 90, 230404 (2003)

# BCS-Bose Superfluidity in 3D

SdM Physics Today 2008



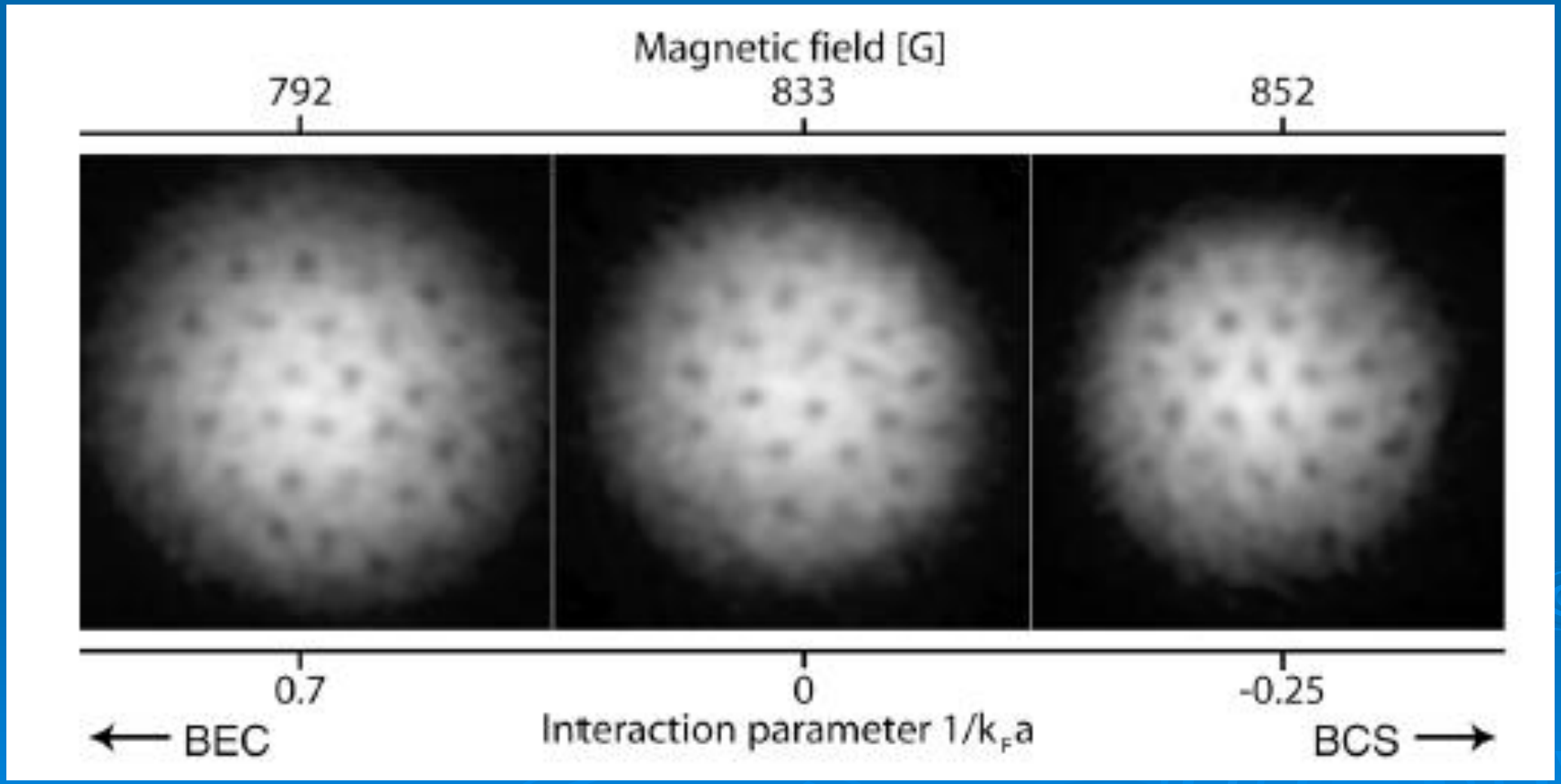

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VOLUME 71, NUMBER 19 PHYSICAL REVIEW LETTERS 8 NOVEMBER 1993

**Crossover from BCS to Bose Superconductivity: Transition Temperature and Time-Dependent Ginzburg-Landau Theory**

C. A. R. Sá de Melo,[1] Mohit Randeria,[2] and Jan R. Engelbrecht[3]

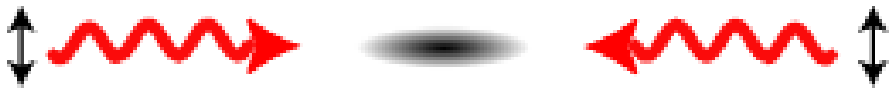
# Vortex Lattice $^{6}$Li (MIT Group)



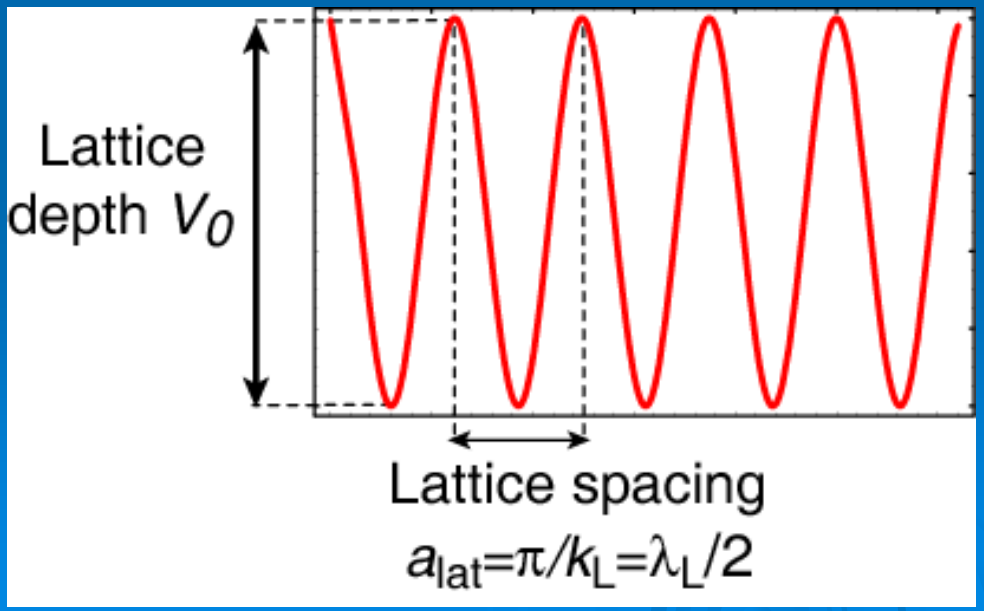
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# Idealized 1D Optical Lattice



$$V(x) = V_0 \sin^2(k_L x)$$




Natural scales: $^{87}$Rb

- Recoil energy:

$$E_R = \frac{h^2}{2m\lambda_L^2}$$

$E_R/h$ ~ 3.2 kHz (150 nK)
@ $\lambda_L$=850 nm

- Lattice spacing:

$a_{lat}=\lambda_L/2$ = 425nm

# Experimental Creation of 2D Fermi Gases in $^6$Li

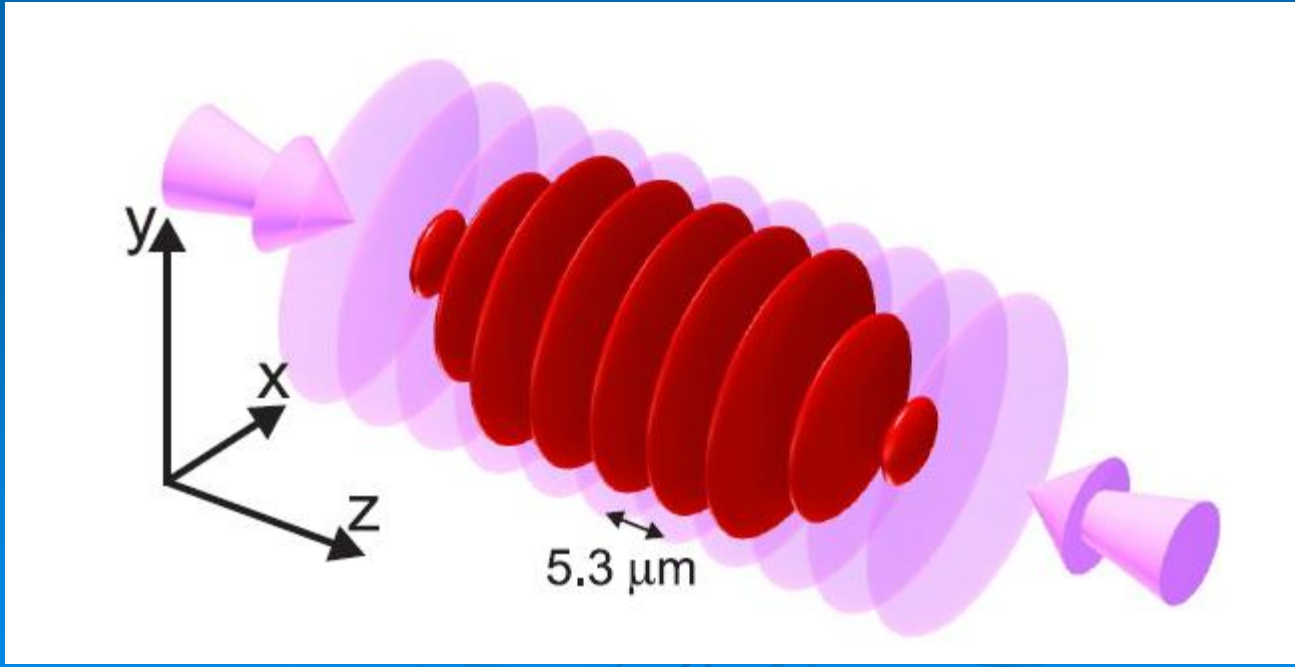
PRL **105,** 030404 (2010) PHYSICAL REVIEW LETTERS week ending 16 JULY 2010

**Observation of a Two-Dimensional Fermi Gas of Atoms**

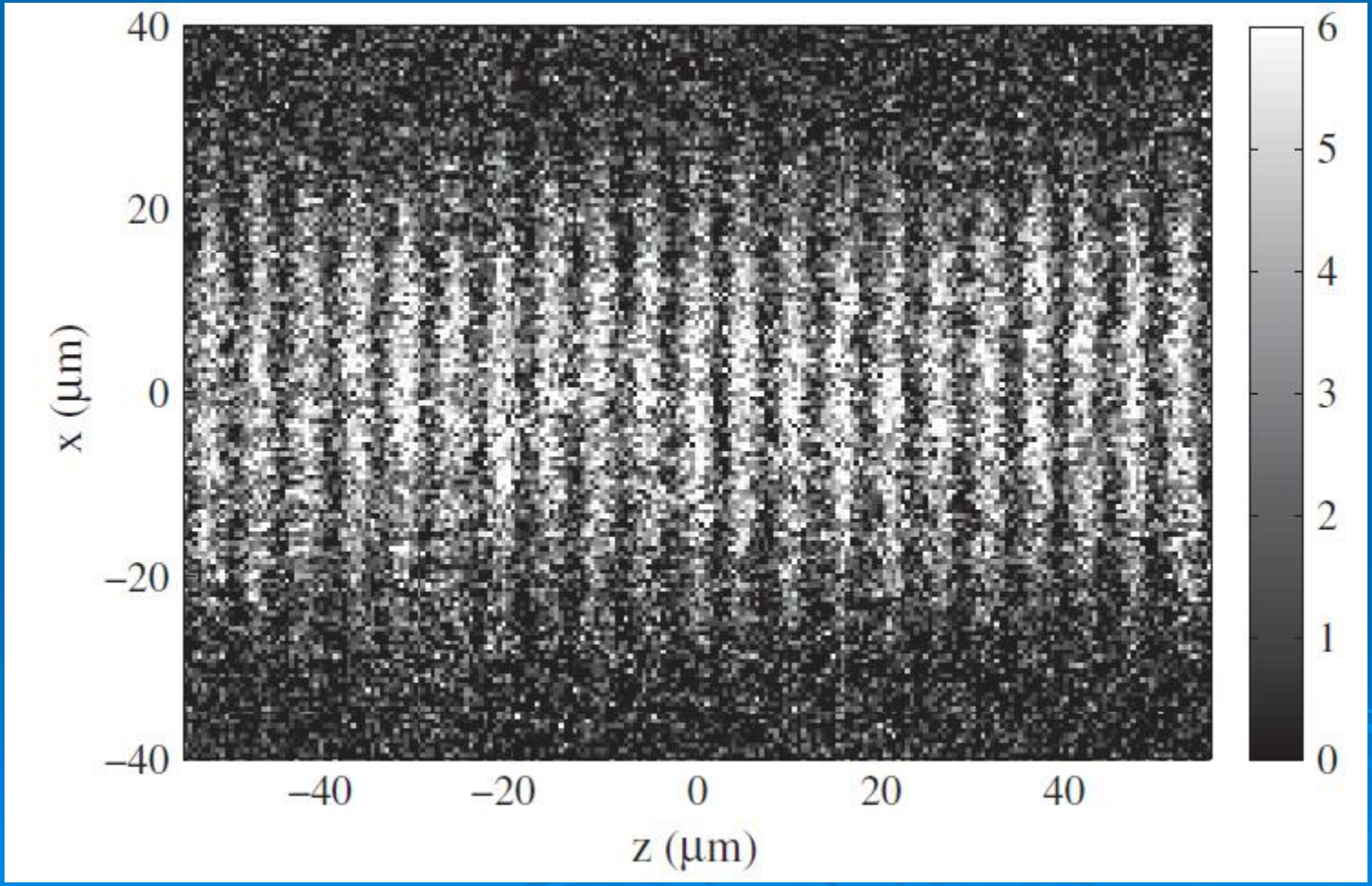
Kirill Martiyanov, Vasiliy Makhalov, and Andrey Turlapov*

*Institute of Applied Physics, Russian Academy of Sciences, ul. Ulyanova 46, Nizhniy Novgorod, 603000, Russia*

(Received 20 May 2010; published 15 July 2010)



# Integrated atom density distribution



# 2D Gases in a Box

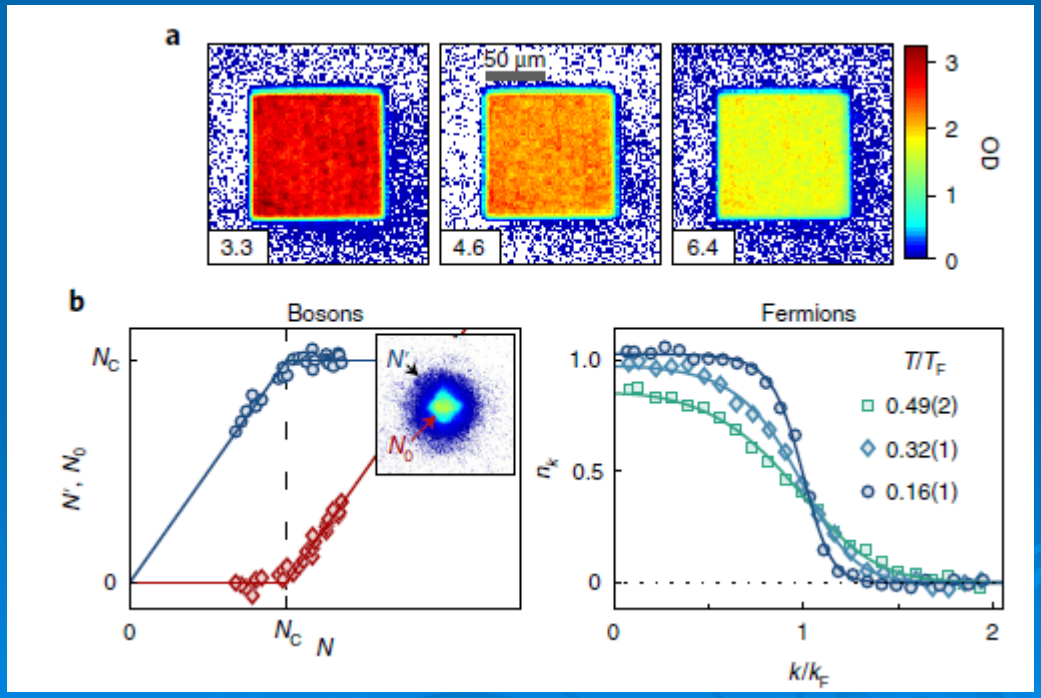
REVIEW ARTICLE | INSIGHT
https://doi.org/10.1038/s41567-021-01403-z

nature physics

Check for updates

## Quantum gases in optical boxes

Nir Navon[1], Robert P. Smith[2] and Zoran Hadzibabic[3]




Use Digital Mirror Devices to create a 2D Box Potential

# Preamble: The 2D XY model

$$H = -J \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j = -J \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j).$$

$$\cos(\theta_i - \theta_j) \sim 1 - \tfrac{1}{2}(\theta_i - \theta_j)^2 \qquad \theta_i - \theta_j = \partial_x \theta$$

$$H = E_0 + \frac{J}{2} \int d\mathbf{r} (\nabla \theta)^2$$

$$E_0 = 2JN$$

# Preamble: The 2D XY model

$$H = E_0 + \frac{J}{2}\int d\vec{r}\,(\vec{\nabla}\theta)^2 \qquad d\vec{r} \equiv d^2r$$

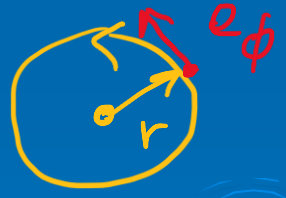
$$\frac{\delta H}{\delta\theta(\mathbf{r})} = 0 \;\Rightarrow\; \nabla^2\theta(\mathbf{r}) = 0.$$

No circulation

$$\theta(\mathbf{r}) = constant.$$

$$\oint \nabla\theta(\mathbf{r})\cdot d\mathbf{l} = 2\pi n.$$

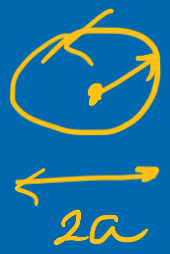
$$\vec{\nabla}\theta = n\,\hat{e}_\phi / r$$




$$\oint_C n\frac{\hat{e}_\phi}{r}\cdot dl = \oint_C n\frac{dl}{r} = n\oint_C d\phi = 2\pi n$$

# Energy to create a vortex

$$
\begin{aligned}
E_{vor} - E_0 &= \frac{J}{2}\int d\mathbf{r}[\nabla\theta(\mathbf{r})]^2 \\
&= \frac{Jn^2}{2}\int_0^{2\pi}\int_a^L r dr \frac{1}{r^2} \\
&= \pi n^2 J \ln(\frac{L}{a}).
\end{aligned}
$$




Grows logarithmically with system size!

# Kosterlitz-Thouless Argument

$$F = E - TS$$

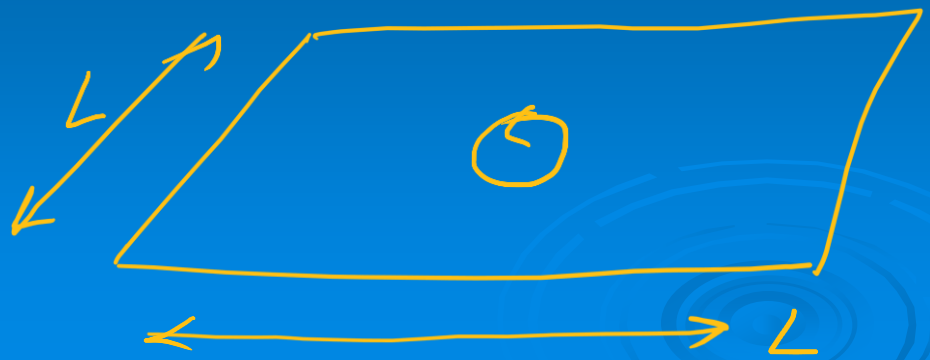
$$E_{v,A} = E_0 + \pi J \ln(L/a)$$

$$F = E_0 + (\pi J - 2k_B T)\ln(L/a).$$




$$S = k_B \ln(L^2/a^2)$$

Entropy



# Kosterlitz-Thouless Argument

$$F = E_0 + (\pi J - 2k_B T)\ln(L/a).$$

Costs too much free energy!

For $T < \pi J/2k_B$ $F \to \infty$ as $L \to \infty$

For $T > \pi J/2k_B$ $F \to -\infty$ as $L \to \infty$

Means that the system can lower its free energy by producing vortices!

# Kosterlitz-Thouless Argument

This means that vortices (and antivortices) must proliferate for

$$k_B T > \frac{\pi}{2} J$$

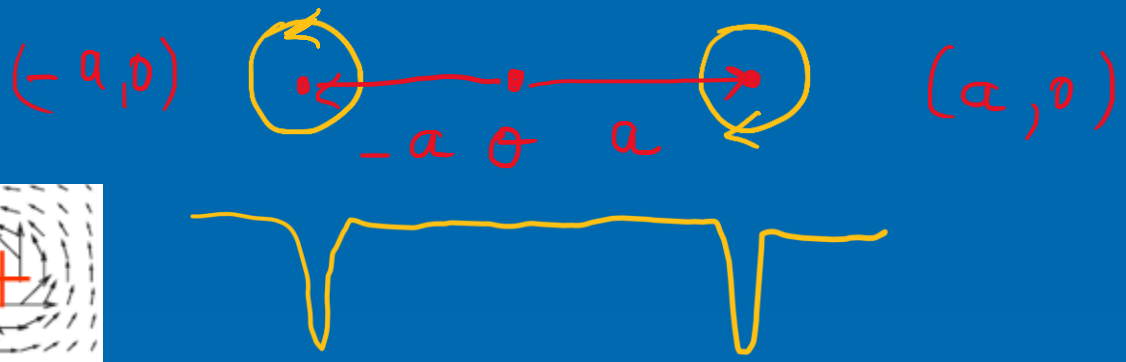
! Since F is lowered !

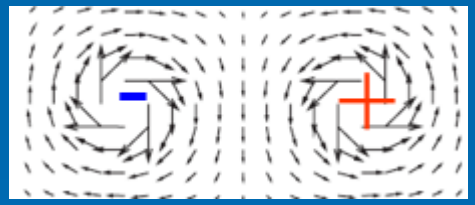
Question: what happens for $k_B T < \frac{\pi}{2} J$?

Answer: Vortices and antivortices must be bound !

# Energy to create a vortex-antivortex pair

$$\oint d\mathbf{l} \cdot \nabla\theta = 2\pi \qquad \oint d\mathbf{l} \cdot \nabla\theta = -2\pi$$



$\theta(\mathbf{r})$ from the vortex–anti-vortex pair is able to cancel out at distances from the centre of the two vortices large compared to the separation $R$ between the vortex and the anti-vortex,

$$\theta_{2vor}(\mathbf{r}) = \text{arctg}\left(\frac{2ay}{a^2 - r^2}\right)$$

$$E_{2vor}(R) = 2E_c + E_1 \ln(R/a)$$

# Renormalization tomorrow

- Renormalization will basically confirm KT argument as a good starting approximation and will provide more!

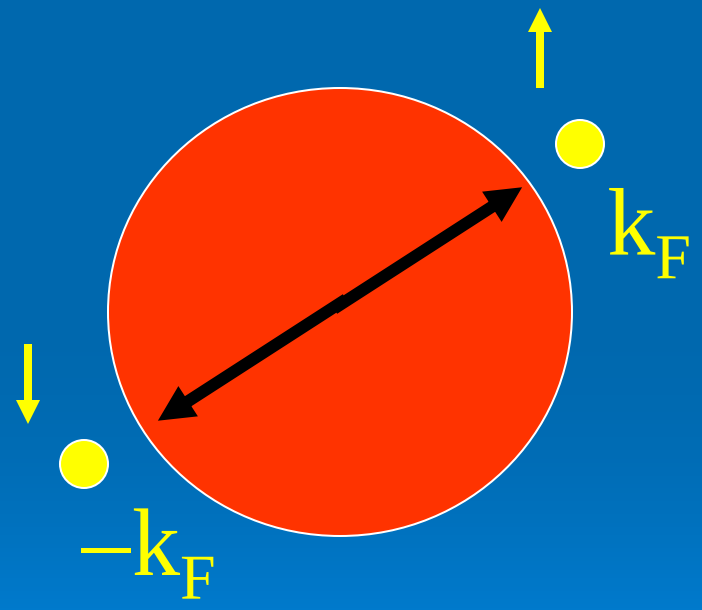
- But today let us rely on the KT argument and apply it to superfluids!

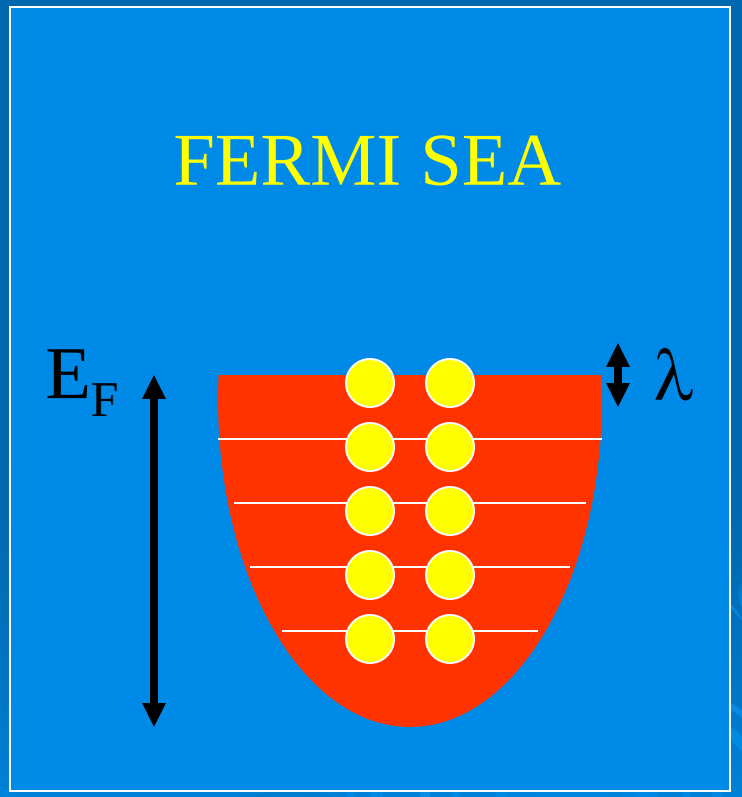
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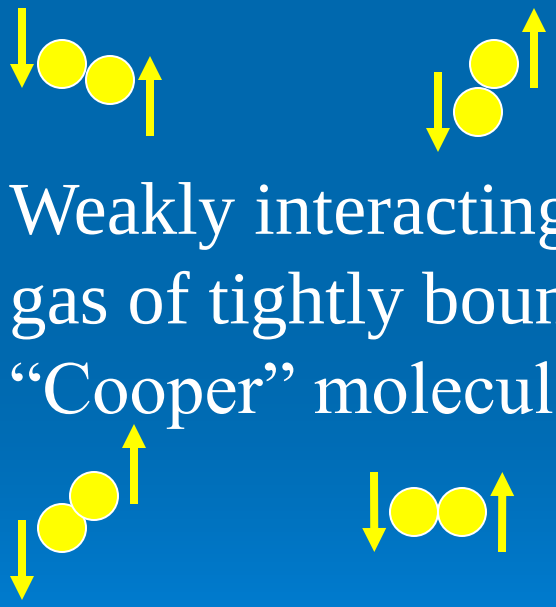
# BCS Pairing ($\lambda << E_F$)

$\mu = E_F > 0$

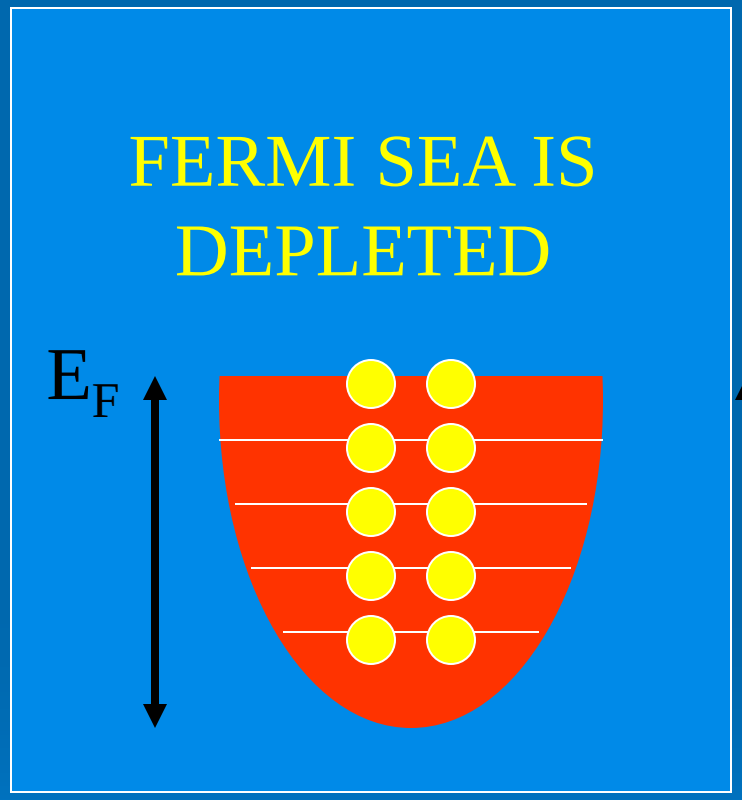


# BEC Pairing ($\lambda >> E_F$)

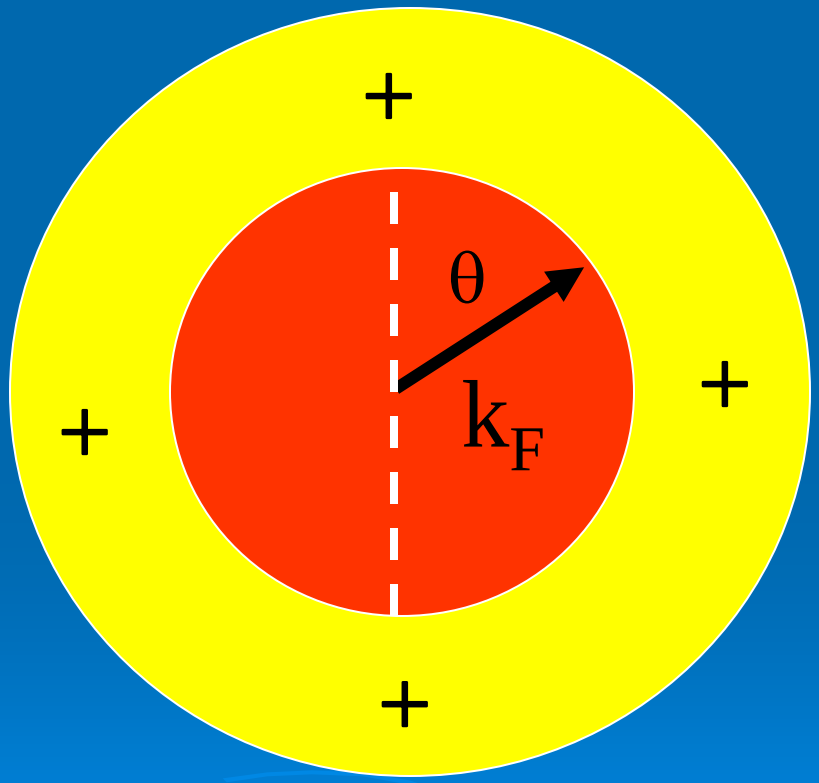


Weakly interacting gas of tightly bound "Cooper" molecules




$\mu = -E_b/2 < 0$

# S-wave Order Parameter (BCS)



# Hamiltonian in 2D

$$\mathcal{H} = \sum_{\mathbf{k},\sigma} \xi_{\mathbf{k}} \psi^{\dagger}_{\mathbf{k}\sigma} \psi_{\mathbf{k}\sigma} + \sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} V_{\mathbf{k}\mathbf{k}'} b^{\dagger}_{\mathbf{k}\mathbf{q}} b_{\mathbf{k}'\mathbf{q}},$$

$$b_{\mathbf{k}\mathbf{q}} = \psi_{-\mathbf{k}+\mathbf{q}/2,\downarrow} \psi_{\mathbf{k}+\mathbf{q}/2,\uparrow}$$

While saddle-point (mean-field) plus Gaussian fluctuations are sufficient to describe the 3D physics of the evolution from BCS to Bose superfluidity, a direct application of the same idea to 2D is problematic and does not capture the BKT physics that is expected for such systems.

# Use functional integral method to obtain effective action

$$Z = \int \mathcal{D}\phi \mathcal{D}\phi^* \exp(-S_{\text{eff}}[\phi, \phi^*]),$$

$$S_{\text{eff}} = \int_0^\beta d\tau \sum_{\mathbf{k}} \left( \frac{|\phi_{\mathbf{k}}(\tau)|^2}{\lambda} + \xi_{\mathbf{k}} \right) - \mathbf{Tr}\left[\ln \mathbf{G}^{-1}_{\mathbf{k},\mathbf{k}'}(\tau)\right].$$

$$\phi_{\mathbf{q}}(\tau) = \Delta_0 \delta_{\mathbf{q},0} + \eta_{\mathbf{q}}(\tau)$$

In real space $\phi(\vec{r},\tau) = |\Delta_0| e^{i\theta(\vec{r},\tau)}$ only phase fluctuations!

# Effective Action

$$S_{\text{eff}} = S_0[\Delta_0] + S_{\text{fluct}}[\eta, \eta^*],$$

$$S_{\text{fluct}} = \frac{1}{2}\int dr[\rho_{ij}\partial_i\theta(r)\partial_j\theta(r) + A(\partial_\tau\theta)^2 - iB\partial_\tau\theta(r)].$$

periodic boundary cond. in $\tau$!

$$\theta(\mathbf{r}, \tau) = \theta_v(\mathbf{r}) + \theta_s(\mathbf{r}, \tau)$$

$$S_{\text{fluct}} = S_v + S_{\text{sw}},$$

# Quantum BKT Effective Action

$$S_v = \frac{1}{2} \int dr \rho_s [\nabla \theta_v(r)]^2$$

$$S_{\mathrm{sw}} = \frac{1}{2} \int dr [\rho_s [\nabla \theta_{\mathrm{sw}}(r)]^2 + A[\partial_\tau \theta_{\mathrm{sw}}(r)]^2$$

$$S_{\mathrm{eff}} = S_0 + S_{\mathrm{sw}} + S_v$$

$$\Omega = \Omega_0 + \Omega_{\mathrm{sw}} + \Omega_v$$

# Self-Consistency Equations

$$\frac{1}{\lambda} = \sum_{\mathbf{k}} \frac{|\Gamma(\mathbf{k})|^2}{2E_{\mathbf{k}}} \tanh\left(\frac{E_{\mathbf{k}}}{2T}\right),$$

$$\frac{1}{\lambda} = \sum_{\mathbf{k}} \frac{|\Gamma(\mathbf{k})|^2}{2\epsilon_{\mathbf{k}} - E_b}.$$

Lippman–Schwinger. (Bound state condition)

Order Parameter Equation

# Self-Consistency Equations

$$N_p = N_0 + N_{\mathrm{sw}}.$$

$$N_0 = -\partial \Omega_0 / \partial \mu$$

$$N_{\mathrm{sw}} = -\partial \Omega_{\mathrm{sw}} / \partial \mu.$$

Number Equation

# Self-Consistency Equations

$$T_{\mathrm{BKT}} = \frac{\pi}{2}\rho_s(\mu, \Delta_0, T_{\mathrm{BKT}}).$$

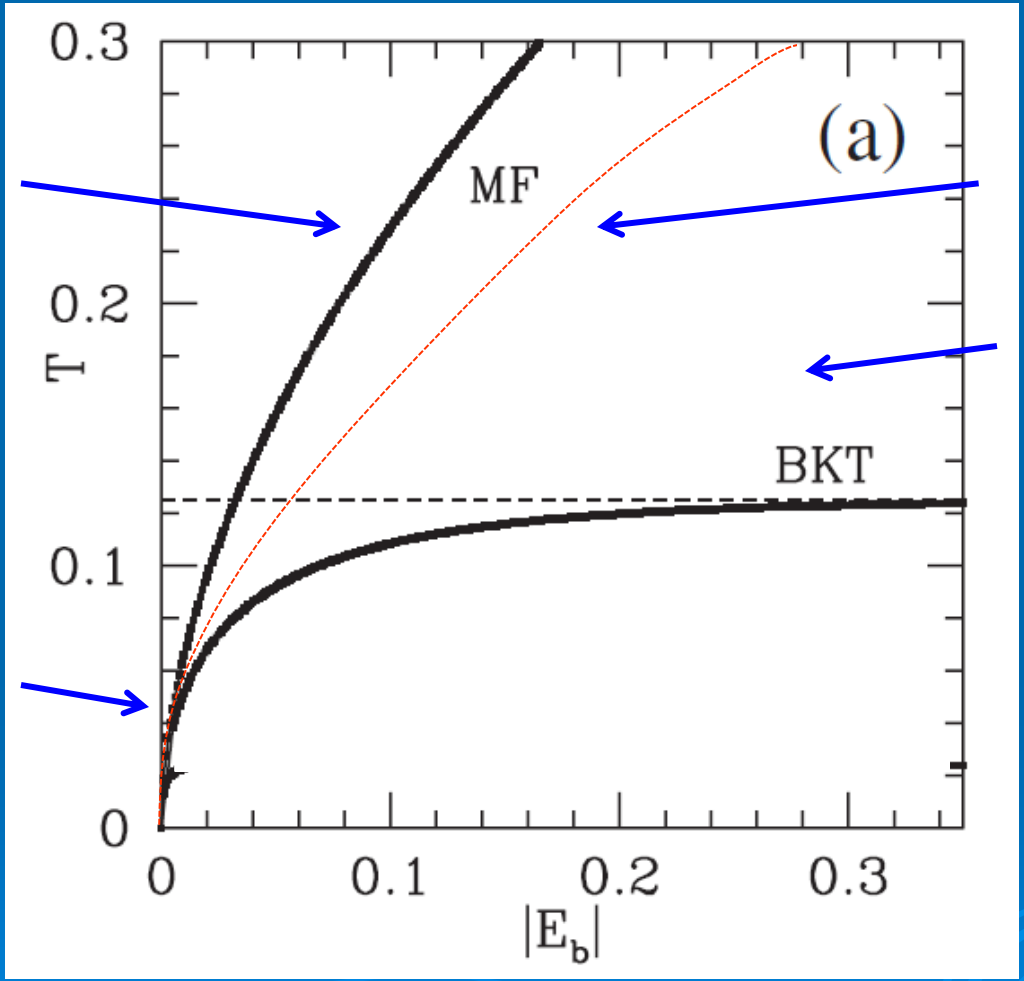
$$\rho_{ij}(\mu, \Delta_0, T) = \frac{1}{4L^2}\sum_{\mathbf{k}}[2n_0(\mathbf{k})\partial_i\partial_j\xi_{\mathbf{k}} - Y_{\mathbf{k}}\partial_i\xi_{\mathbf{k}}\partial_j\xi_{\mathbf{k}}],$$

BKT condition

# Critical Temperature

Pairing Temperature

Fermi Liquid




Gaussian fluctuations

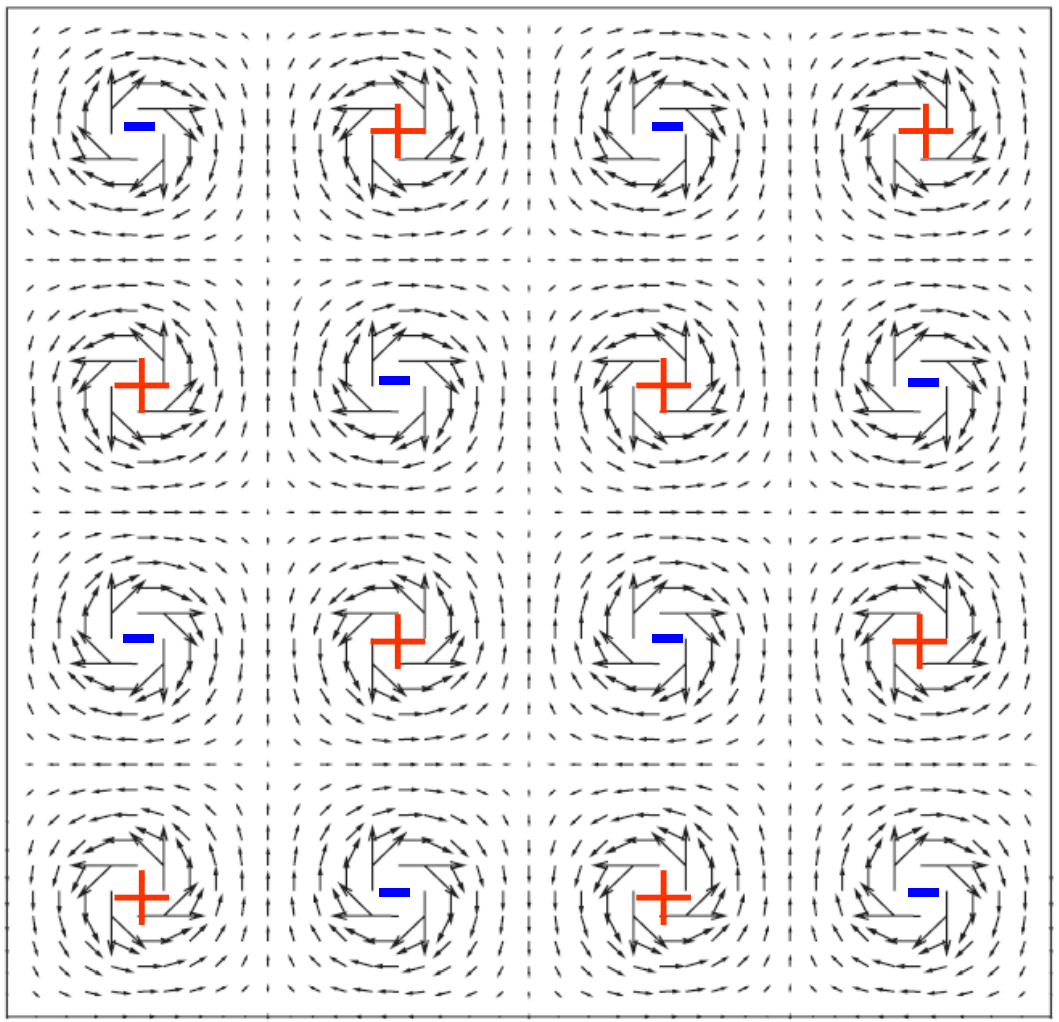
Bose Liquid

0.125

BCS-Bose Superfluidity in 2D

# Vortex-Antivortex Lattice

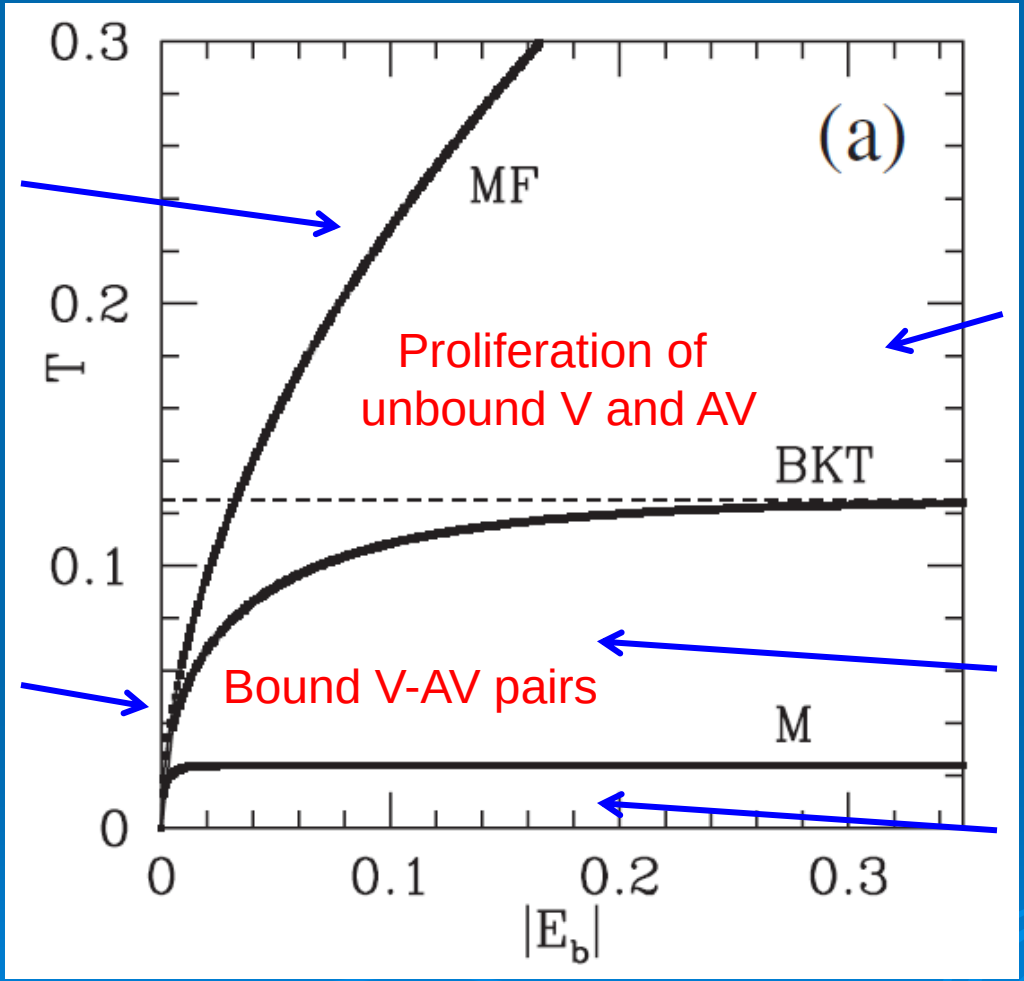
$$TS_v = H_v = (2\pi)\frac{\rho_s}{2}\sum_{i\neq j} n_i n_j G(\mathbf{r}_i - \mathbf{r}'_j) + \sum_i E_c n_i^2,$$



$$\theta_{\mathrm{VA}}(x, y) = \arctan\left(\frac{2ay}{a^2 - x^2 - y^2}\right).$$

# Conclusions in Pictures

Pairing Temperature




Bose Liquid

0.125

Fermi Liquid

V-AV Liquid

V-AV Lattice

BCS-Bose Superfluidity in 2D

# Outline

- Introduction: from BCS to Bose superfluidity.
- Creating ultra-cold fermions in the flatland.
- BCS to Bose superfluidity in two-dimensions.
- Conclusions.

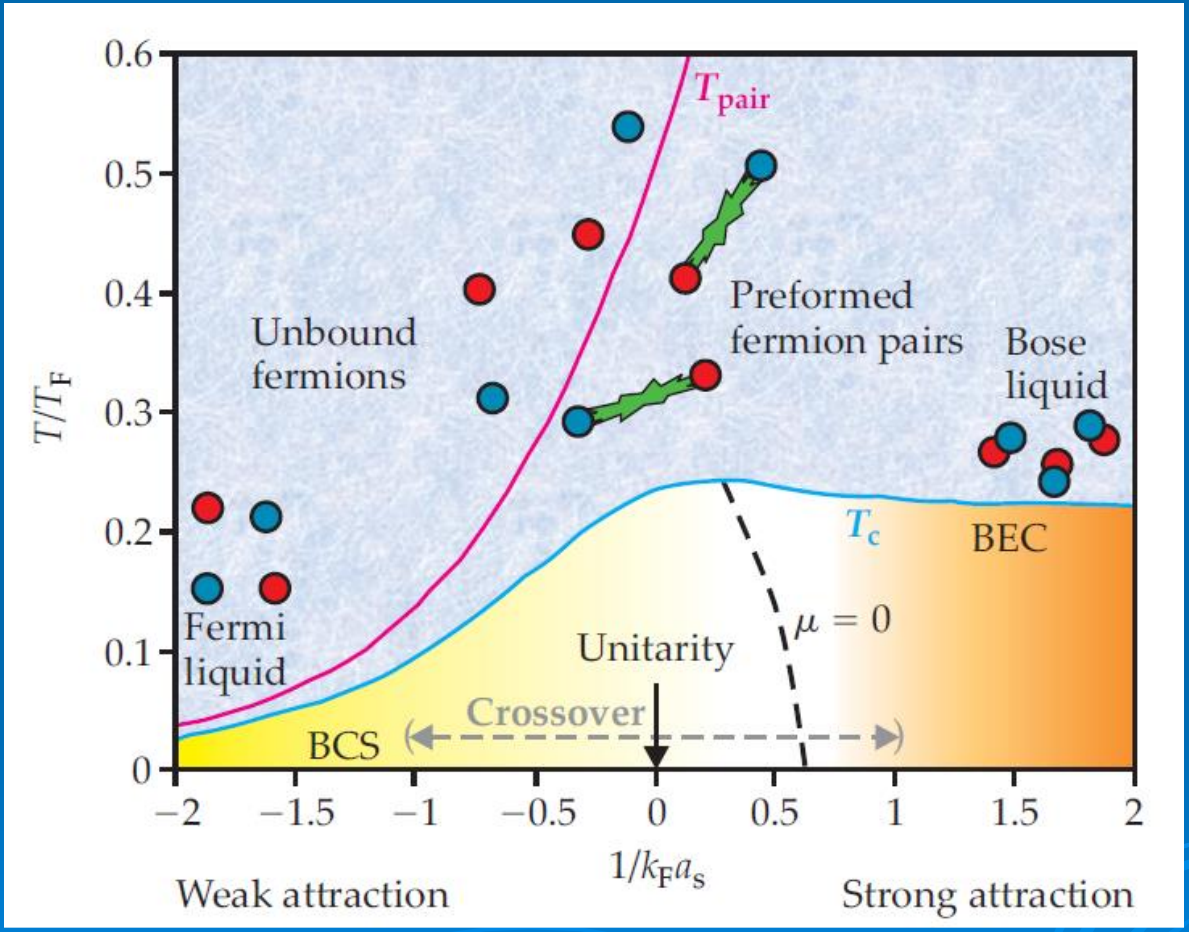
# Conclusions in Words

- The evolution from BCS to Bose superfluidity in 3D s-wave systems is analytic, i.e., a crossover.

- The evolution from BCS to Bose superfluidity in 2D s-wave systems is non-analytic, i.e., there is a phase transition, where the BKT state at low temperature may form a vortex-antivortex lattice.

# Conclusions in Pictures

SdM Physics Today 2008



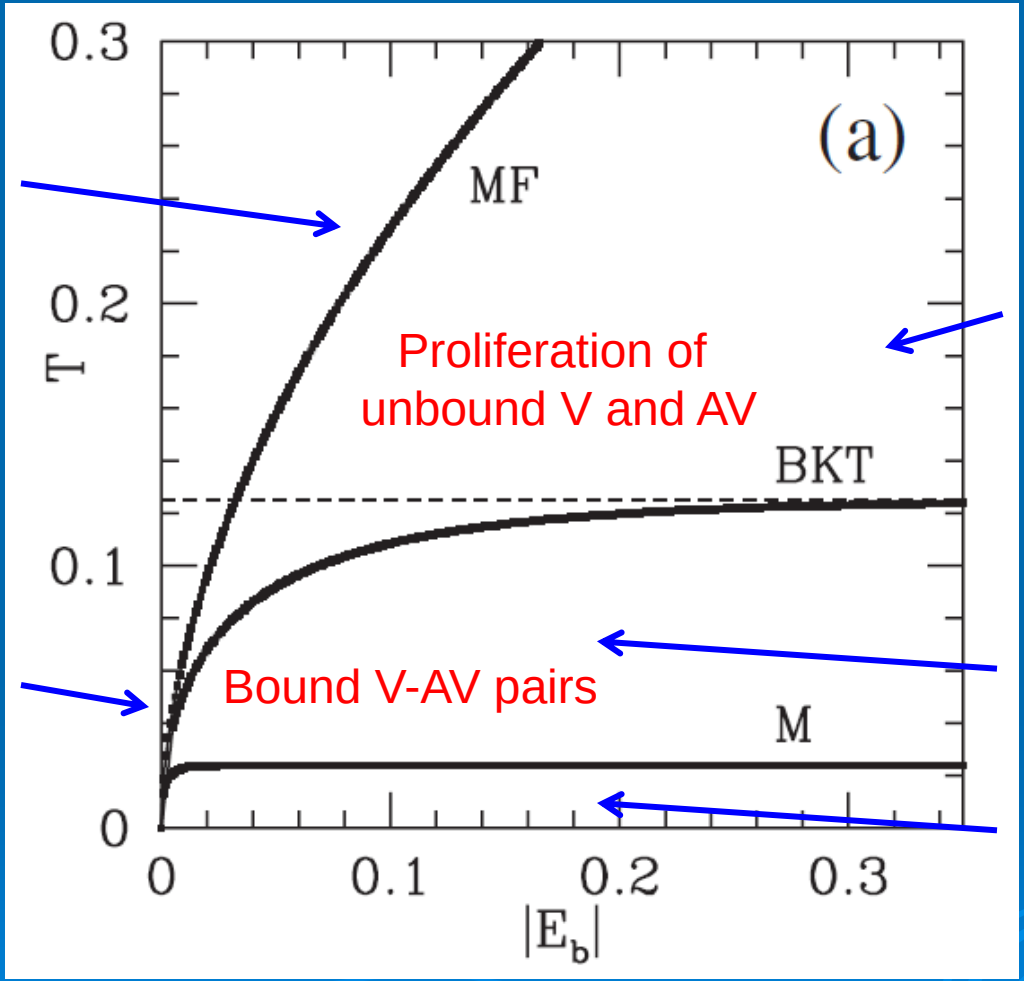

0.218

BCS-Bose Superfluidity in 3D

# Conclusions in Pictures

Pairing Temperature

Fermi Liquid



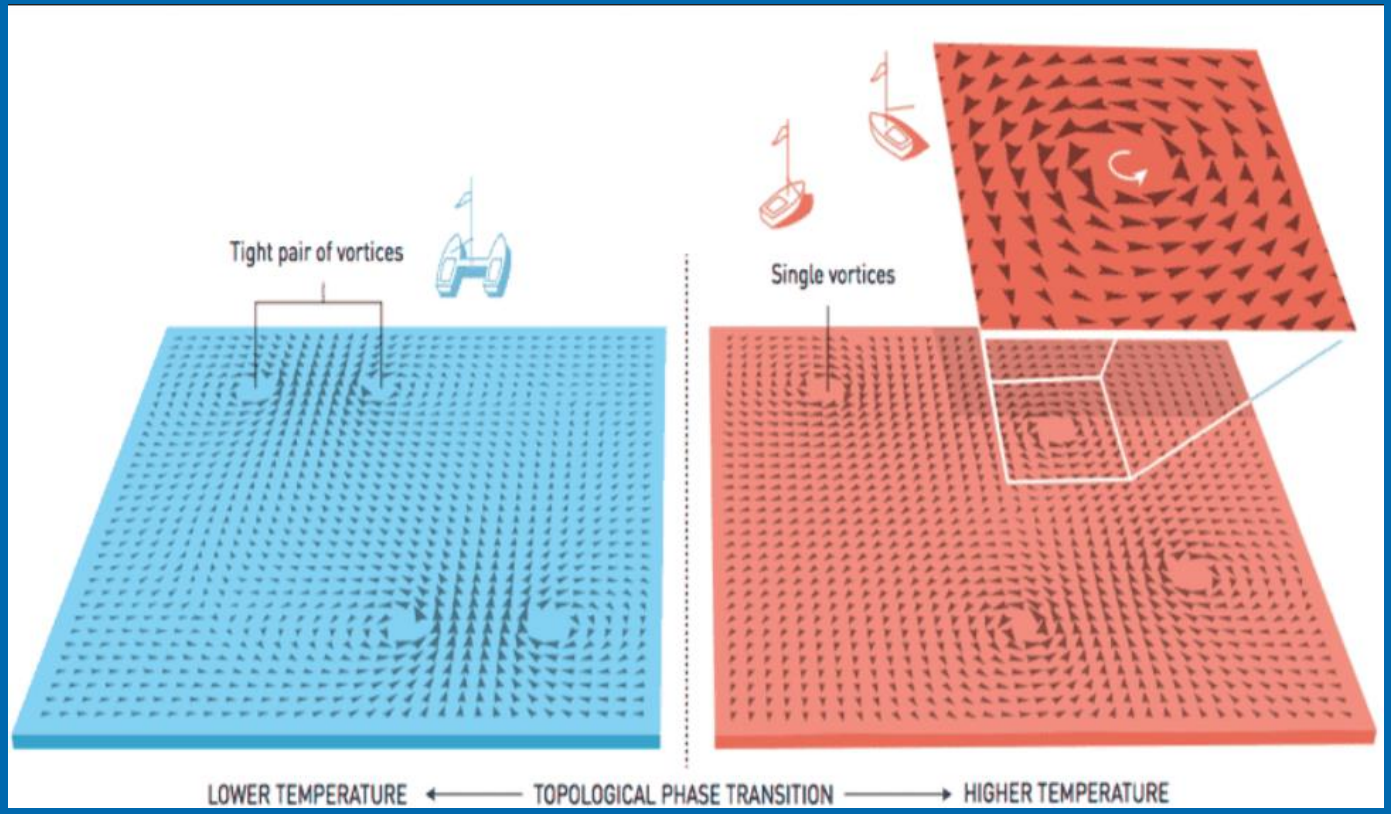

Bose Liquid

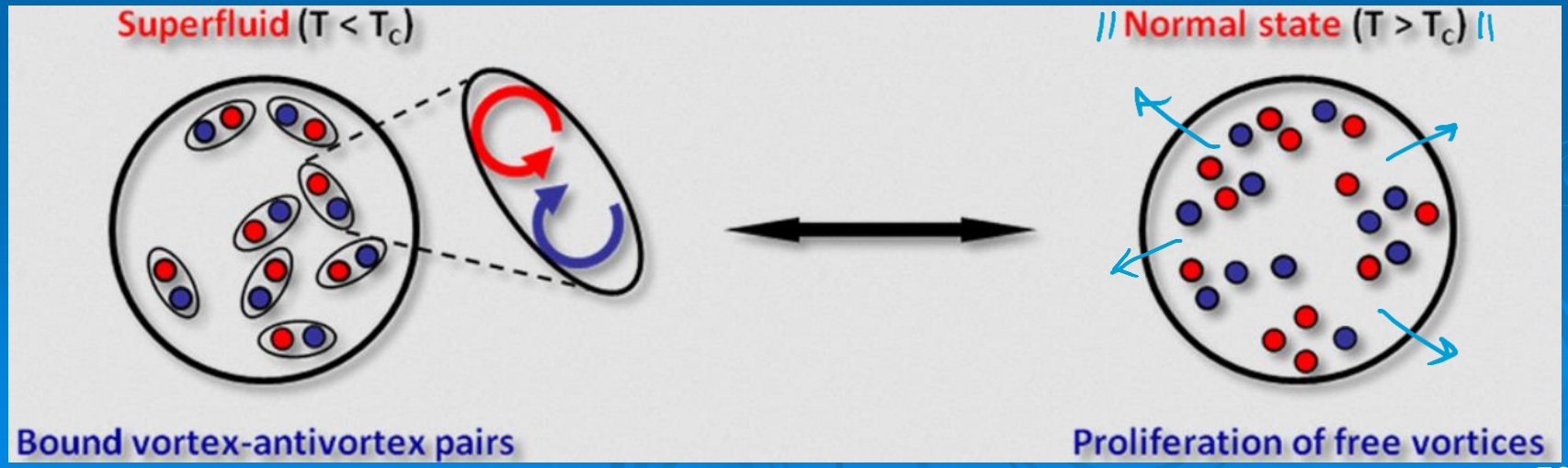
0.125

V-AV Liquid

V-AV Lattice

BCS-Bose Superfluidity in 2D


Tight pair of vortices
Single vortices
LOWER TEMPERATURE
TOPOLOGICAL PHASE TRANSITION
HIGHER TEMPERATURE


Superfluid (T < Tc)
Normal state (T > Tc)
Bound vortex-antivortex pairs
Proliferation of free vortices

# End of Lecture I