From BCS to Bose Superfluidity in Two-dimensional Fermi Gases:

Lecture II: Renormalization Group and Tighter Upper Bounds on Critical Temperature

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Outline

- Is there an upper bound for the T_c of 2D superfluids (continuum or lattice)?
- Can the Renormalization Group provide a tighter upper bound on T_c in 2D superfluids?
- Conclusions

Conclusions in Words

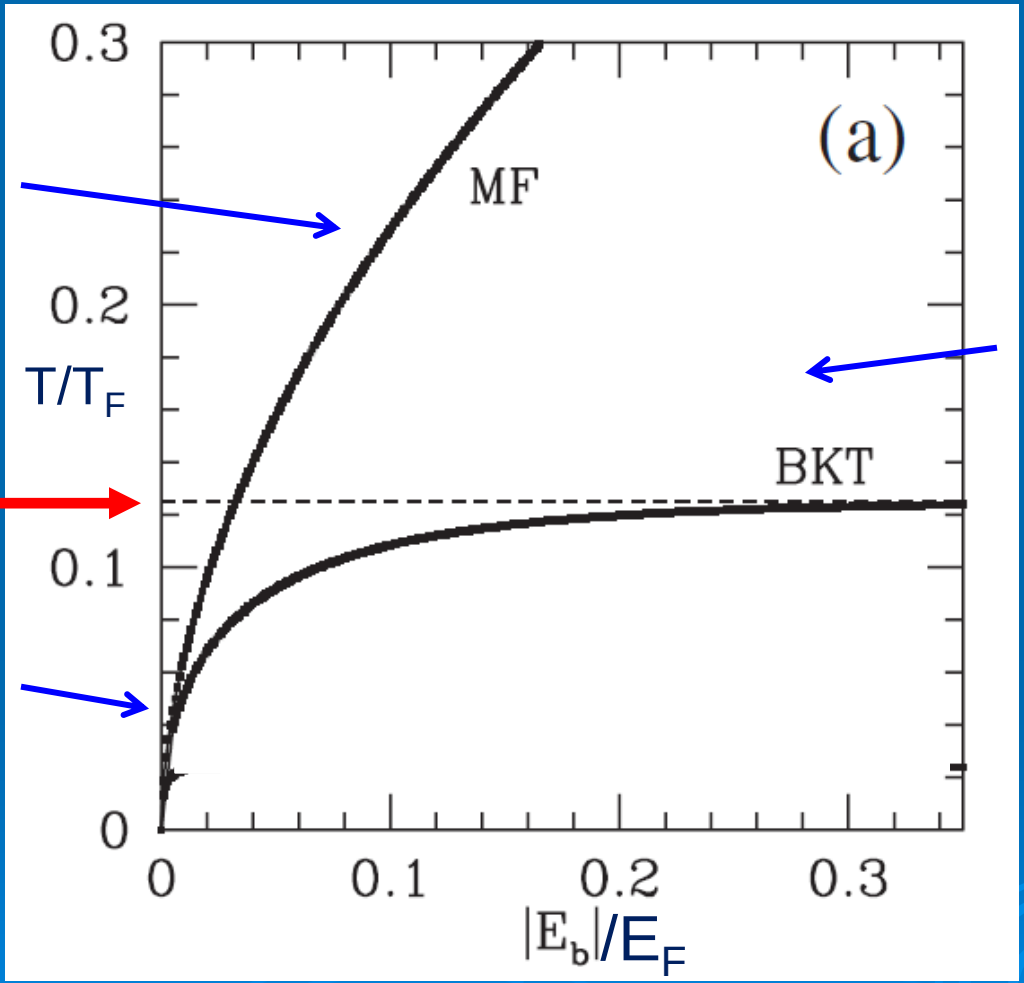
- Yes, there is an upper bound for the critical temperature of one-band two-dimensional superfluids.
- A tighter upper bound is provided by the Renormalization Group and the renormalized superfluid density.

Upper Bound for Critical Temperature in 2D

Pairing Temperature

Upper Bound $T/T_F = 1/8$

Fermi Liquid



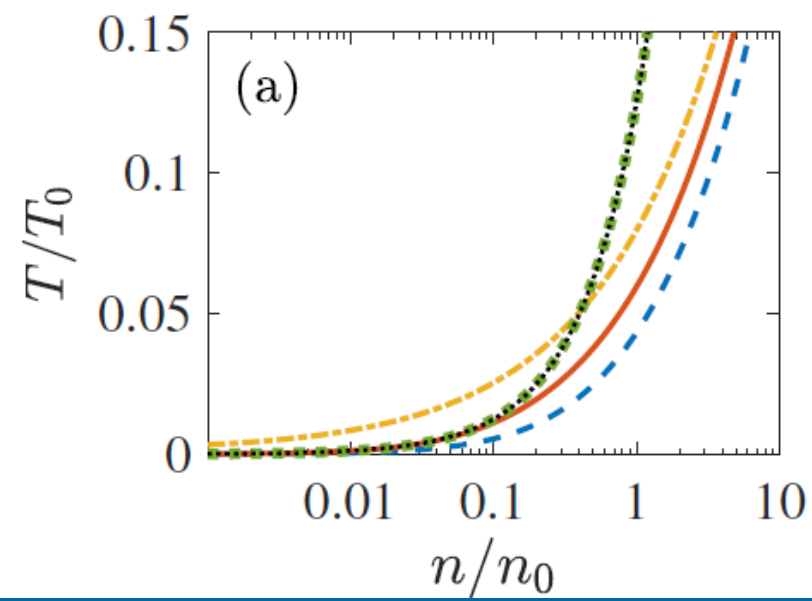
Bose Liquid

0.125

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Conclusions in Pictures



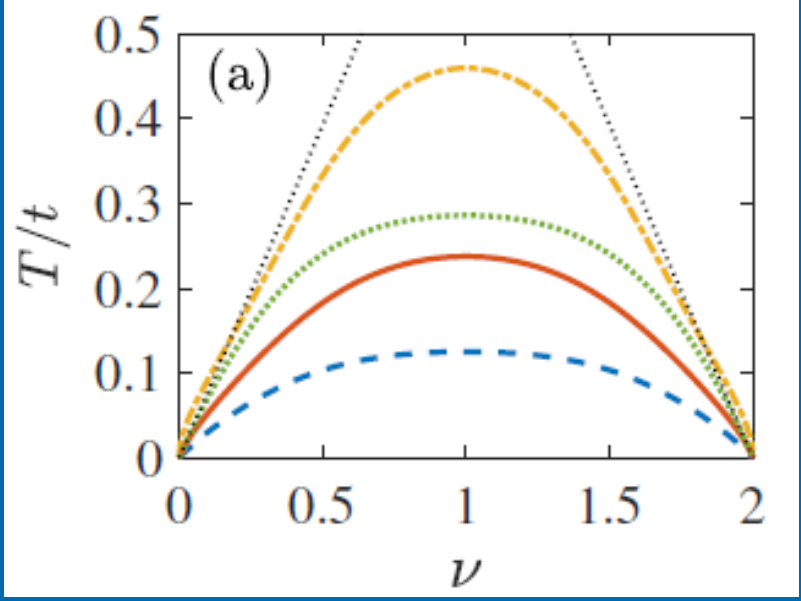


Table 1. List of temperatures obtained by different methods.

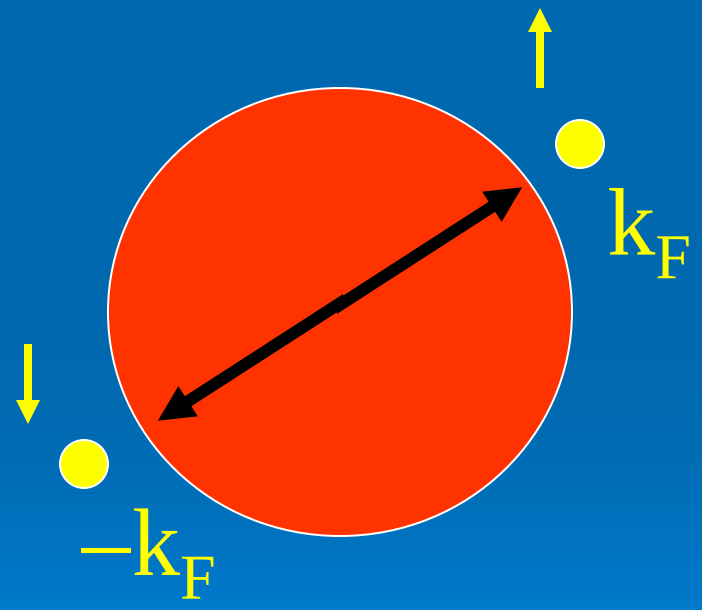
T_c	Description $T_c^{\mathrm{sup}} \leq T_c^{\theta} \leq T_c^{\mathrm{up0}} \leq T_c^{\mathrm{up1}}$
T_c^{sup}	Critical temperature supremum
T_c^{θ}	Phase-fluctuation critical temperature
T_c^{up0}	Upper bound set by superfluid density $\rho_s(T=0)$
T_c^{up1}	Upper bound set by kinetic energy
T_c^{mf}	Mean-field critical temperature
$T_c^{\theta\mathrm{mod}}$	Modulus and phase-fluctuation T_c

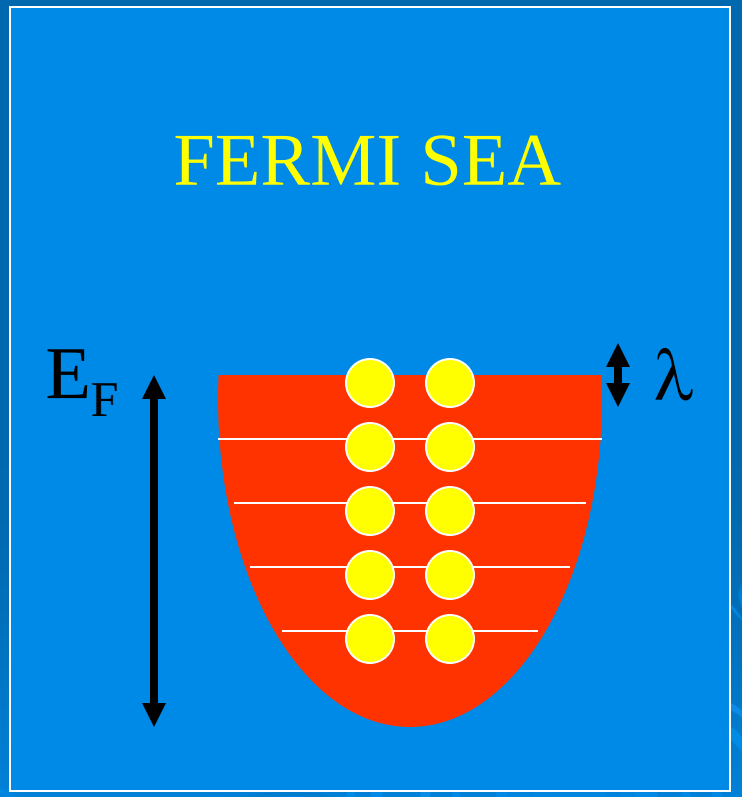
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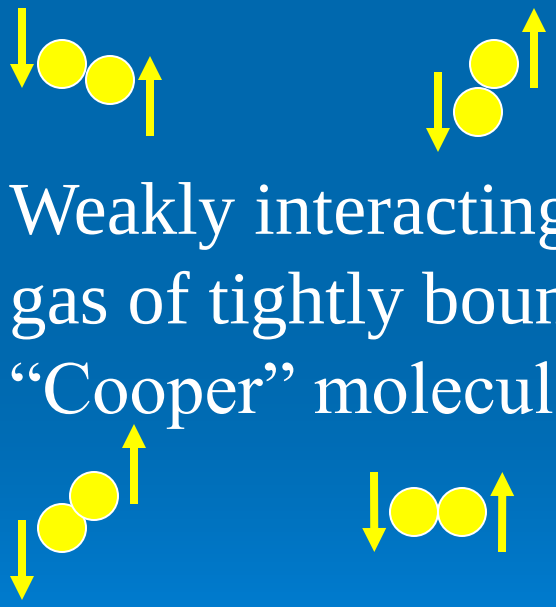
BCS Pairing ($\lambda << E_F$)

$\mu = E_F > 0$

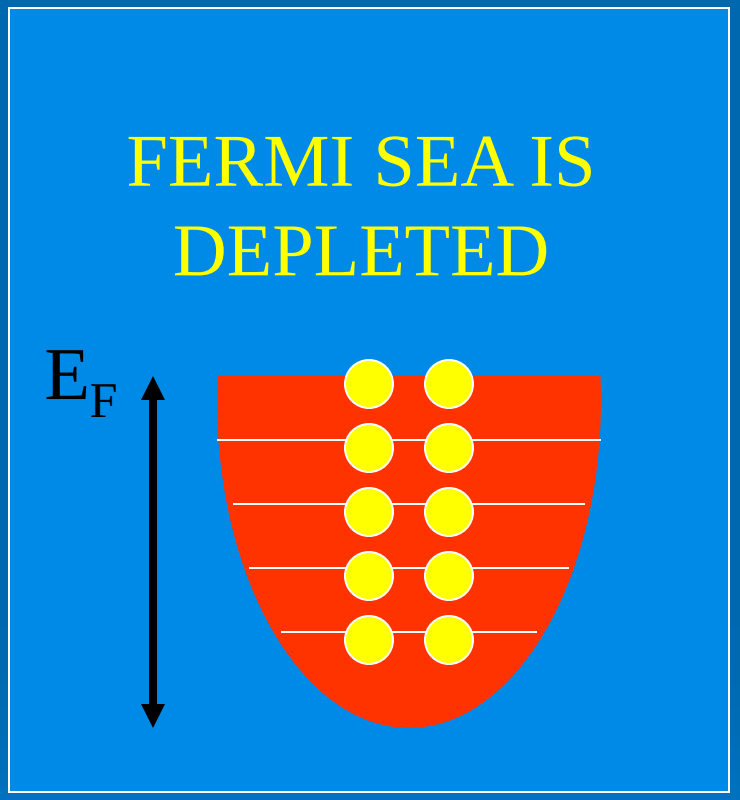




BEC Pairing ($\lambda >> E_F$)



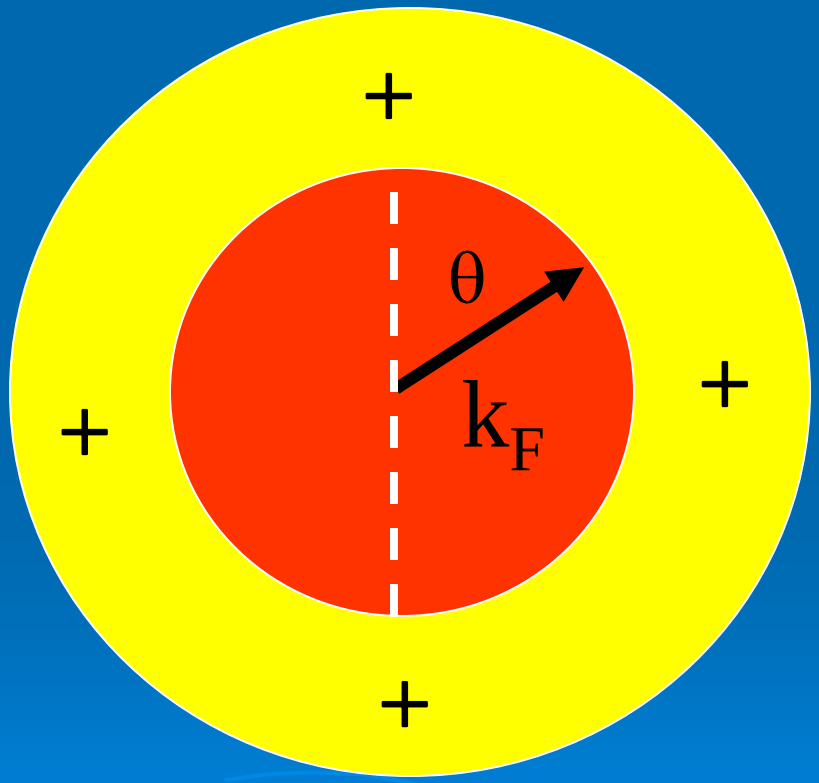
Weakly interacting gas of tightly bound "Cooper" molecules



$\mu = -E_b/2 < 0$



S-wave Order Parameter (BCS)



Hamiltonian in 2D

$$\mathcal{H} = \sum_{\mathbf{k},\sigma} \xi_{\mathbf{k}} \psi^{\dagger}_{\mathbf{k}\sigma} \psi_{\mathbf{k}\sigma} + \sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} V_{\mathbf{k}\mathbf{k}'} b^{\dagger}_{\mathbf{k}\mathbf{q}} b_{\mathbf{k}'\mathbf{q}},$$

$$b_{\mathbf{k}\mathbf{q}} = \psi_{-\mathbf{k}+\mathbf{q}/2,\downarrow} \psi_{\mathbf{k}+\mathbf{q}/2,\uparrow}$$

While saddle-point (mean-field) plus Gaussian fluctuations are sufficient to describe the 3D physics of the evolution from BCS to Bose superfluidity, a direct application of the same idea to 2D is problematic and does not capture the BKT physics that is expected for such systems.

Use functional integral method to obtain effective action

$$Z = \int \mathcal{D}\phi \mathcal{D}\phi^* \exp(-S_{\text{eff}}[\phi, \phi^*]),$$

$$S_{\text{eff}} = \int_0^\beta d\tau \sum_{\mathbf{k}} \left(\frac{|\phi_{\mathbf{k}}(\tau)|^2}{\lambda} + \xi_{\mathbf{k}} \right) - \mathbf{Tr}\left[\ln \mathbf{G}^{-1}_{\mathbf{k},\mathbf{k}'}(\tau)\right].$$

$$\phi_{\mathbf{q}}(\tau) = \Delta_0 \delta_{\mathbf{q},0} + \eta_{\mathbf{q}}(\tau)$$

In real space $\phi(\vec{r},\tau) = |\Delta_0| e^{i\theta(\vec{r},\tau)}$ only phase fluctuations!

Effective Action

$$S_{\text{eff}} = S_0[\Delta_0] + S_{\text{fluct}}[\eta, \eta^*],$$

$$S_{\text{fluct}} = \frac{1}{2}\int dr[\rho_{ij}\partial_i\theta(r)\partial_j\theta(r) + A(\partial_\tau\theta)^2 - iB\partial_\tau\theta(r)].$$

periodic boundary cond. in τ!

$$\theta(\mathbf{r}, \tau) = \theta_v(\mathbf{r}) + \theta_s(\mathbf{r}, \tau)$$

$$S_{\text{fluct}} = S_v + S_{\text{sw}},$$

Quantum BKT Effective Action

$$S_v = \frac{1}{2} \int dr \rho_s [\nabla \theta_v(r)]^2$$

$$S_{\mathrm{sw}} = \frac{1}{2} \int dr [\rho_s [\nabla \theta_{\mathrm{sw}}(r)]^2 + A[\partial_\tau \theta_{\mathrm{sw}}(r)]^2$$

$$S_{\mathrm{eff}} = S_0 + S_{\mathrm{sw}} + S_v$$

$$\Omega = \Omega_0 + \Omega_{\mathrm{sw}} + \Omega_v$$

Self-Consistency Equations

$$\frac{1}{\lambda} = \sum_{\mathbf{k}} \frac{|\Gamma(\mathbf{k})|^2}{2E_{\mathbf{k}}} \tanh\left(\frac{E_{\mathbf{k}}}{2T}\right),$$

$$\frac{1}{\lambda} = \sum_{\mathbf{k}} \frac{|\Gamma(\mathbf{k})|^2}{2\epsilon_{\mathbf{k}} - E_b}.$$

Lippman–Schwinger. (Bound state condition)

Order Parameter Equation

Self-Consistency Equations

$$N_p = N_0 + N_{\mathrm{sw}}.$$

$$N_0 = -\partial \Omega_0 / \partial \mu$$

$$N_{\mathrm{sw}} = -\partial \Omega_{\mathrm{sw}} / \partial \mu.$$

Number Equation

Self-Consistency Equations

$$T_{\mathrm{BKT}} = \frac{\pi}{2}\rho_s(\mu, \Delta_0, T_{\mathrm{BKT}}).$$

BKT condition

$$\rho_{ij}(\mu, \Delta_0, T) = \frac{1}{4L^2}\sum_{\mathbf{k}}[2n_0(\mathbf{k})\partial_i\partial_j\xi_{\mathbf{k}} - Y_{\mathbf{k}}\partial_i\xi_{\mathbf{k}}\partial_j\xi_{\mathbf{k}}],$$

If $\rho_{ij} = \rho_s \delta_{ij}$

$$\rho_s \le \frac{1}{4L^2} N \frac{1}{m} = \frac{n}{4m}$$

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If $\rho_{ij} = \rho_s \delta_{ij}$

$$\rho_s \le \frac{1}{4L^2} N \frac{1}{m} = \frac{n}{4m}$$

Upper bound set by kinetic energy

$$T_c^{\mathrm{up1}} = \frac{\pi}{2}\left(\frac{n}{4m}\right) = \frac{\pi n}{8m} = \frac{\varepsilon_F}{8}$$

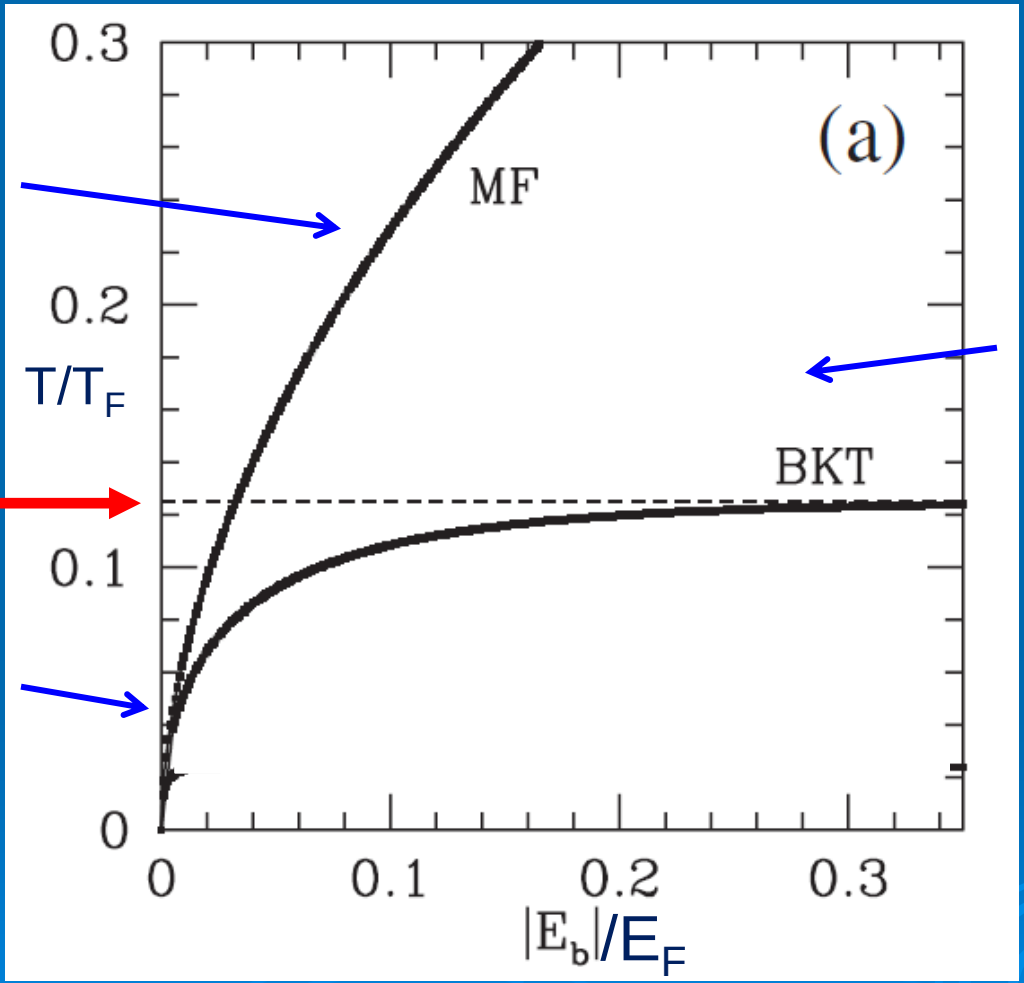
Galilean invariant system

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Renormalization Group and tighter upper bounds for T_c in 2D

Tighter upper bounds on the critical temperature of two-dimensional superfluids and superconductors from the BCS to the Bose regime

Tingting Shi[1,2,3,4], Wei Zhang[1,3,†], C. A. R. Sá de Melo[2,‡]

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$$\mathcal{H}(\mathbf{r}) = \mathcal{H}_{\rm K}(\mathbf{r}) + \mathcal{H}_{\rm I}(\mathbf{r})$$

$$\mathcal{H}_{\rm K}(\mathbf{r}) = \sum_s \psi_s^\dagger(\mathbf{r}) \left[-\frac{\nabla^2}{2m} \right] \psi_s(\mathbf{r})$$

2D Continuum

$$\mathcal{H}_{\rm I}(\mathbf{r}) = \int d^2\mathbf{r}' V(\mathbf{r},\mathbf{r}') \psi_\uparrow^\dagger(\mathbf{r}) \psi_\downarrow^\dagger(\mathbf{r}') \psi_\downarrow(\mathbf{r}') \psi_\uparrow(\mathbf{r})$$

2D Square Lattice

$$H = -t \sum_{\langle ij \rangle, s} \psi_{is}^\dagger \psi_{js} - U \sum_{is} \hat{n}_{i\uparrow} \hat{n}_{i\downarrow} - \sum_{i<j, ss'} V_{ij} \hat{n}_{is} \hat{n}_{js'},$$

Effective Action

We introduce the chemical potential μ and the order parameter Δ for superfluidity in terms of its modulus $|\Delta|$ and phase θ. By fixing the modulus $|\Delta|$ and including full phase fluctutations, the effective action is

$$S_{\text{eff}} = S_{\text{sp}}\left(|\Delta|\right) + S_{\text{ph}}(|\Delta|, \theta).$$

$$S_{\text{sp}} = \sum_{\mathbf{k}} \left[\frac{(\xi_{\mathbf{k}} - E_{\mathbf{k}})}{T} - 2\ln\left(1 + e^{-E_{\mathbf{k}}/T}\right) \right] + \frac{|\Delta|^2}{TV_s},$$

$$S_{\text{ph}} = \frac{1}{2}\int dr \left\{ \sum_{ij} \rho_{ij}\, \partial_i\theta(r)\partial_j\theta(r) + \kappa_s \left[\partial_\tau\theta(r)\right]^2 \right\}$$

Bare Superfluid Density

$$\rho_{ij} = \frac{1}{4L^2}\sum_{\mathbf{k}}\left[2n_{\rm sp}(\mathbf{k})\partial_i\partial_j\xi_{\mathbf{k}} - Y_{\mathbf{k}}\partial_i\xi_{\mathbf{k}}\partial_j\xi_{\mathbf{k}}\right]$$

$$\rho_{ij} = \rho_s\delta_{ij}$$

2D Continuum

$$\rho_s = \frac{1}{4mL^2}\sum_{\mathbf{k}}\left[2n_{\rm sp}(\mathbf{k}) - (k_x^2/m)Y_{\mathbf{k}}\right]$$

2D Square Lattice

$$\rho_s = \frac{ta^2}{L^2}\sum_{\mathbf{k}}\left[\cos(k_x a)n_{\rm sp}(\mathbf{k}) - t\sin^2(k_x a)Y_{\mathbf{k}}\right]$$

Compressibility ~

$$\kappa_s = \frac{1}{4L^2}\sum_{\mathbf{k}}\left[\frac{|\Delta_{\mathbf{k}}|^2}{E_{\mathbf{k}}^3}\tanh\left(\frac{E_{\mathbf{k}}}{2T}\right) + \frac{\xi_{\mathbf{k}}^2}{E_{\mathbf{k}}^2}Y_{\mathbf{k}}\right]$$

Longitudinal and Transverse Velocities

$$S_{\rm ph} = S_c + S_v.$$

$$S_c = \frac{1}{2}\int dr\,\left[\rho_s\left[\nabla\theta_c(r)\right]^2 + \kappa_s\left[\partial_\tau\theta_c(r)\right]^2\right]$$

$$S_v = \frac{1}{2T}\int d^2\mathbf{r}\,\rho_s\left[\nabla\theta_v(\mathbf{r})\right]^2$$

$$\mathbf{v}_t = \nabla\theta_v(\mathbf{r})$$

$$\nabla\cdot\mathbf{v}_t(\mathbf{r}) = 0.$$

$$\nabla\times\mathbf{v}_t(\mathbf{r}) = 2\pi\hat{\mathbf{z}}n_v(\mathbf{r})$$

$$n_v(\mathbf{r}) = \textstyle\sum_i n_i\delta(\mathbf{r}-\mathbf{r}_i)$$

Vortex Action

$$S_v = \frac{\rho_s}{2T}(2\pi)^2 \int\int d^2\mathbf{r} d^2\mathbf{r}' n_v(\mathbf{r}) G(\mathbf{r}-\mathbf{r}') n_v(\mathbf{r}'),$$

where $G(\mathbf{r}_i - \mathbf{r}_j)$ is the dimensionless interaction potential between the vortices $n_v(\mathbf{r})$ and $n_v(\mathbf{r}')$, satisfying Poisson's equation $\nabla^2_{\mathbf{r}} G(\mathbf{r}-\mathbf{r}') = 0$. The formal solution to this differential equation is $G(\mathbf{r}-\mathbf{r}') = \int d\mathbf{k} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} / \left[(2\pi)^2 |\mathbf{k}|^2\right]$, with lower (π/L) and upper (π/d) cutoffs to regularize the action. Here, L is the linear system size and d is the minimum separation between vortices, leading to $G(\mathbf{r}-\mathbf{r}') = \left[\ln(L/d) - \ln(|\mathbf{r}-\mathbf{r}'|/d) + C\right]/2\pi$, where C is a constant. In the thermodynamic limit, the system size $L \to \infty$, and topological charge neutrality $\sum_i n_i = 0$ leads to

$$S_v = -2\pi \frac{\rho_s}{2T} \sum_{i \neq j} n_i n_j \ln\left[\frac{|\mathbf{r}-\mathbf{r}'|}{d}\right] + \sum_i \frac{E_c}{T} n_i^2, \qquad (19)$$

$$K(0) = \rho_s / T$$

Self-consistent equations needed

$$\Omega = \Omega_{\rm sp} + \Omega_{\rm ph}$$

$$\Omega_{\rm ph} = \Omega_v + \Omega_c.$$

$$\Omega_c = \Omega_{\rm cm} + \Omega_{\rm zp},$$

$$S_{\rm sp} = \sum_{\rm k} \left[\frac{(\xi_{\rm k} - E_{\rm k})}{T} - 2\ln\left(1 + e^{-E_{\rm k}/T}\right) \right] + \frac{|\Delta|^2}{TV_s},$$

$\Rightarrow$ Order Parameter Eq.

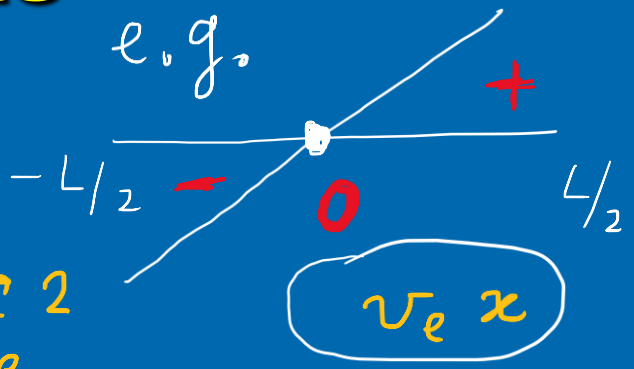
$$n = n_{\rm sp} + n_{\rm cm} + n_{\rm zp} + n_v$$

$\Rightarrow$ number equation

$$S_v = -2\pi \frac{\rho_s}{2T} \sum_{i \neq j} n_i n_j \ln\left[\frac{|\mathbf{r} - \mathbf{r}'|}{d}\right] + \sum_i \frac{E_c}{T} n_i^2,$$

$\Rightarrow$ Nelson-Kosterlitz condition

Compute Free Energy via Phase Twists



$$\theta(\vec{r}) = \theta_0(\vec{r}) + \vec{v}_e \cdot \vec{r}$$

$$\Omega(\vec{v}_{ex}) - \Omega(0) = \frac{1}{2} V \rho_s^R \vec{v}_e^{\,2}$$

K K_0

$$\frac{\rho_s^R}{T} = \frac{\rho_s}{T}\left[\frac{1}{1+\frac{\rho_s}{2T}\chi_v}\right] \approx \frac{\rho_s}{T}\left[1 - \frac{\rho_s}{2T}\chi_v\right]$$

screening

χ_v

$$\rho_s^R = \rho_s - \frac{\rho_s^2}{2T}\lim_{\mathbf{q}\to 0}\frac{\langle n_v(\mathbf{q}) n_v(-\mathbf{q})\rangle}{|\mathbf{q}|^2}$$

Screened superfluid density and vortex fugacity

$$K^{-1} = K_0^{-1}\left[1 + \frac{K_0}{2}\chi_v\right]; \quad K^{-1} = K_0^{-1} + \frac{\chi_v}{2}$$

$$K^{-1}(r) = K_0^{-1} + 4\pi^3 y_0 \int_a^r \frac{dr'}{a}\left(\frac{r'}{a}\right)^{3-2\pi f(r')}$$

$y_0 = e^{-\beta E_c}$ vortex fugacity

Define scale dependent fugacity

$$y(r) = y_0\left(\frac{r}{a}\right)^{(2-\pi f(r))}$$

RG relations

$$K^{-1}(r) = K_0^{-1} + 4\pi^3 \int_a^r \frac{dr'}{r'} y^2(r')$$

$$y(r) = y_0 \exp\left[2\ln(r/a) - \pi \int_a^r dr' \frac{K(r')}{r'}\right]$$

$$\frac{E(r)}{T} = 2\pi \int_a^r dr' \frac{K(r')}{r'} \equiv 2\pi f(r) \ln(r/a)$$

Vortex pair energy

force between charges

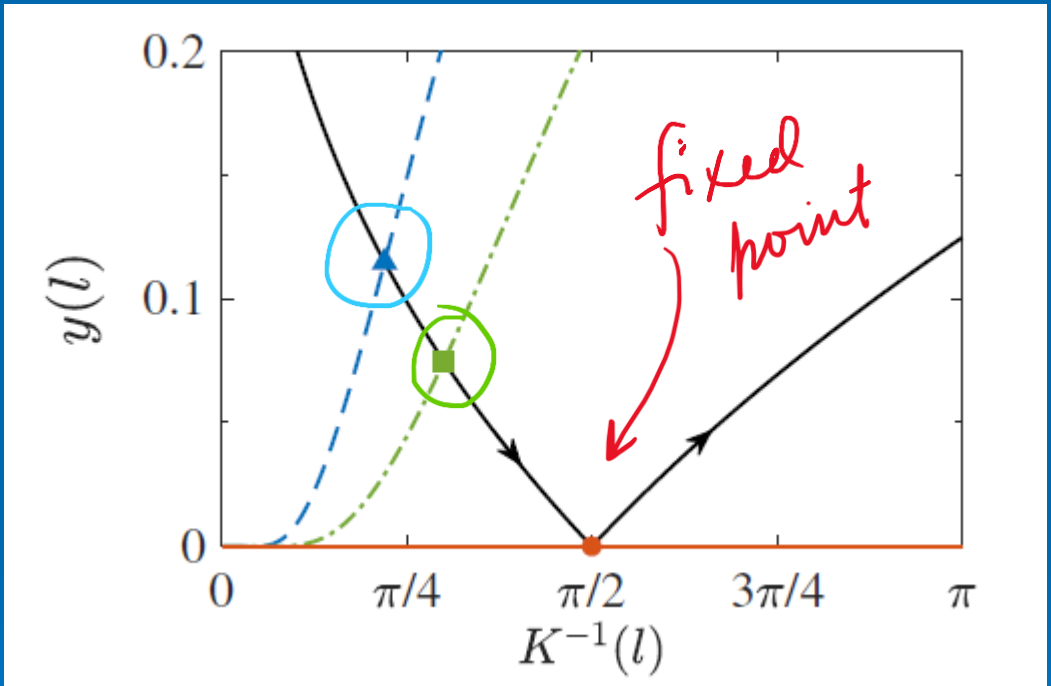
Renormalization Flow

$$\frac{d\tilde{K}^{-1}(\ell)}{d\ell} = 4\pi^3 \tilde{y}^2(\ell) + O(\tilde{y}^3)$$

$$\frac{d\tilde{y}(\ell)}{d\ell} = [2 - \pi K(\ell)]\tilde{y}(\ell) + O(\tilde{y}^2)$$

$\ell = \ln r$

$E_c = 1.5\ ps$ $E_c = 2.45\ ps$



$$y^2(l) - \frac{1}{2\pi^3}\left[\frac{2}{K(l)} + \pi \ln K(l)\right] = A$$

solution

Critical Point (Nelson-Kosterlitz relation)

$$(K_*^{-1}, y_*) = (\pi/2, 0)$$

$$K_c^{-1} \leq \pi/2$$

$$T_c^\theta/\rho_s = K_c^{-1}$$

$$\rho_s^R = \rho_s(2/\pi K_c).$$

$$\rho_s^R \leq \rho_s$$

Renormalized superfluid density always less or equal to bare superfluid density!

The equality between ρ_s^R and ρ_s occurs only when $E_c \to \infty$.

$$T_c^\theta = \frac{\pi}{2}\rho_s^R(\mu, |\Delta|, T_c^\theta)$$

$$T_c^\theta = \pi\rho_s^R/2 \leq \pi\rho_s/2.$$

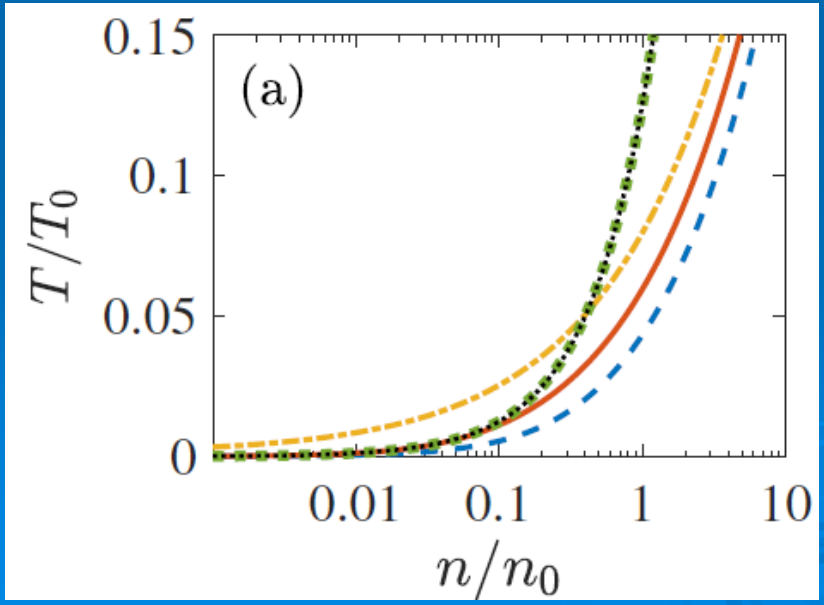
Tighter bound on critical temperature

$$T_c^\theta = \frac{\pi}{2}\rho_s^R(\mu, |\Delta|, T_c^\theta)$$

$$T_c^\theta = \pi\rho_s^R/2 \leq \pi\rho_s/2.$$

The renormalized (screened) superfluid density is always lower than the bare one, and thus it always provides a tighter bound for T_c.

Continuum case



$$T_c^{\rm sup} \leq T_c^\theta \leq T_c^{\rm up0} \leq T_c^{\rm up1}$$

Table 1. List of temperatures obtained by different methods.

T_c	Description
$T_c^{\rm sup}$	Critical temperature supremum
T_c^θ	Phase-fluctuation critical temperature
$T_c^{\rm up0}$	Upper bound set by superfluid density $\rho_s(T=0)$
$T_c^{\rm up1}$	Upper bound set by kinetic energy
$T_c^{\rm mf}$	Mean-field critical temperature
$T_c^{\theta\rm mod}$	Modulus and phase-fluctuation T_c

Square Lattice (Attractive Hubbard Model)

$$T_c^{\mathrm{sup}} \leq T_c^{\theta} \leq T_c^{\mathrm{up0}} \leq T_c^{\mathrm{up1}}$$

$E_c = 1.5\,\rho_s \qquad E_c \to \infty$

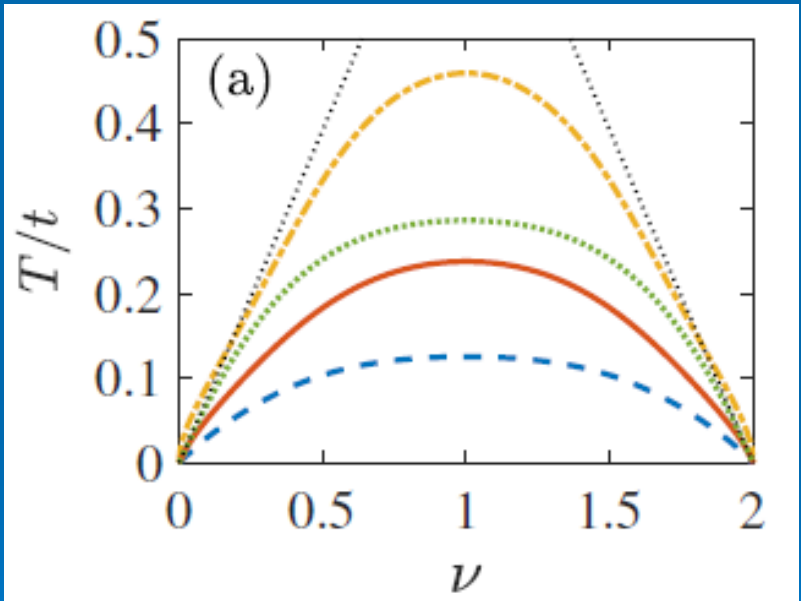


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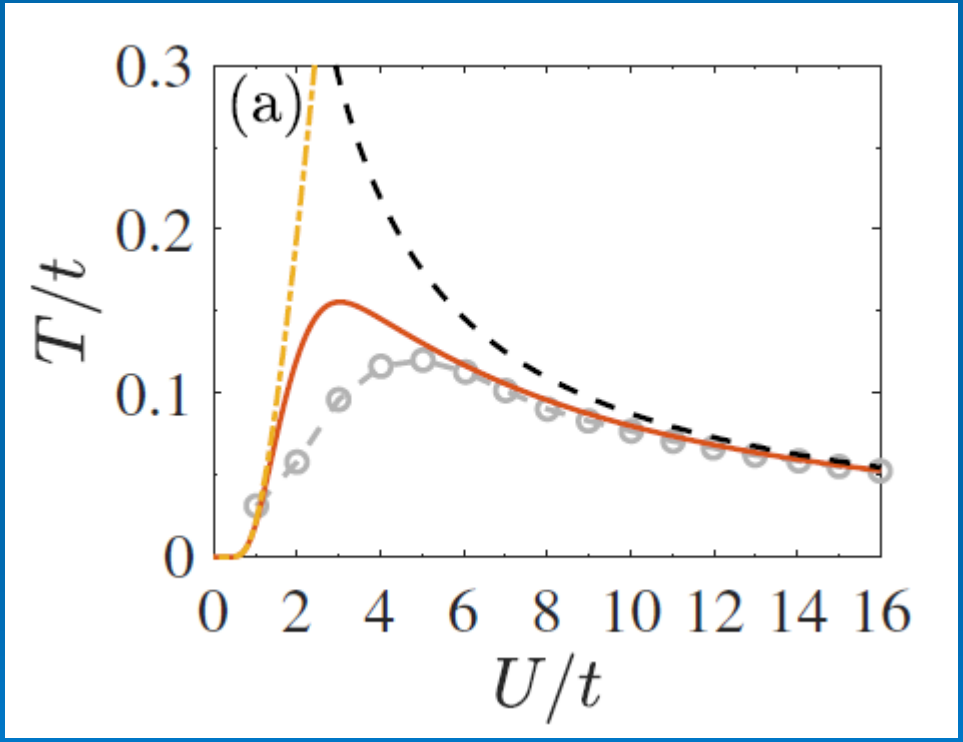
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T_c^{up1}	Upper bound set by kinetic energy
T_c^{mf}	Mean-field critical temperature
$T_c^{\theta\mathrm{mod}}$	Modulus and phase-fluctuation T_c

$$T_c^{\mathrm{up1}} = t\frac{\pi}{4}\min\{\nu, (2-\nu)\}.$$

Modulus fluctuations and comparison to QMC

$E_c = 2.22\ \rho_s$

$K_c^{-1} = 0.89$



$$T_c^{\theta}(U/t \gg 1) = K_c^{-1}\left[\nu(2-\nu)\right] t^2/U.$$

Word of Caution

Finally, our results have a few important implications for the measurements of T_c in 2D one-band superconductors with non-retarded interactions and without disorder. First, the vortex core energy $E_c = \alpha\rho_s$, where ρ_s is the bare superfluid density, must have an α function that depends on density n (or chemical potential μ) as the system evolves from the BCS to the Bose limit. This is in sharp contrast to 2D neutral superfluids, that can be viewed as extreme type-II superconductors with infinite Ginzburg-Landau parameter κ, where $\alpha = 2.45$ is independent of n (or μ) throughout the BCS to Bose evolution. Therefore $\alpha = 2.45$ cannot be blindly used for 2D superconductors. Second, given that the renormalized superfluid density ρ_s^R can be obtained experimentally from penetration depth measurements, and that the critical temperature T_c can be measured resistively, if the measured T_c exceeds the phase fluctuation upper bound $T_c^\theta = \pi\rho_s^R/2$ then a non-BKT mechanism must be invoked.

Outline

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- Can the Renormalization Group provide a tighter upper bound on T_c in 2D superfluids?
- Conclusions

Conclusions in Words

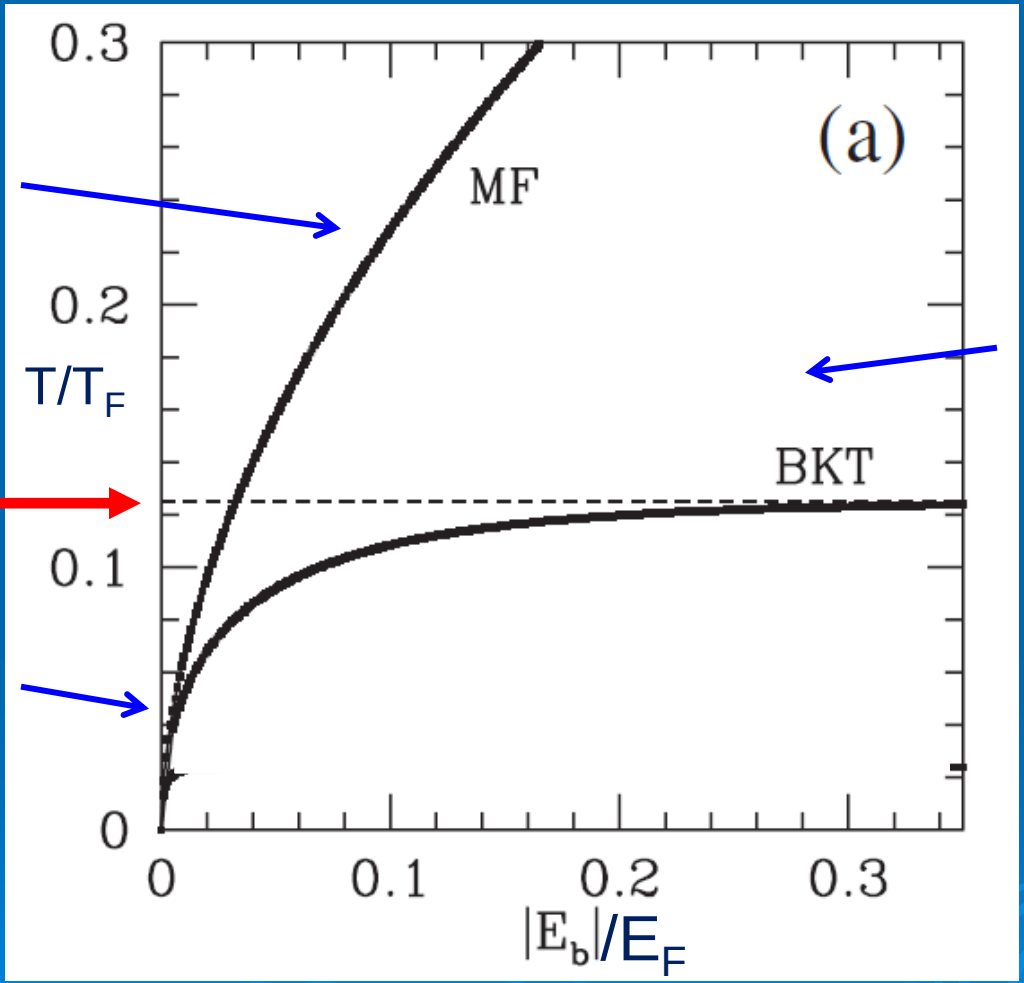
- Yes, there is an upper bound for the critical temperature of one-band two-dimensional superfluids.
- A tighter upper bound is provided by the Renormalization Group and the renormalized superfluid density.

Upper Bound for Critical Temperature in 2D

Pairing Temperature

Upper Bound $T/T_F = 1/8$

Fermi Liquid



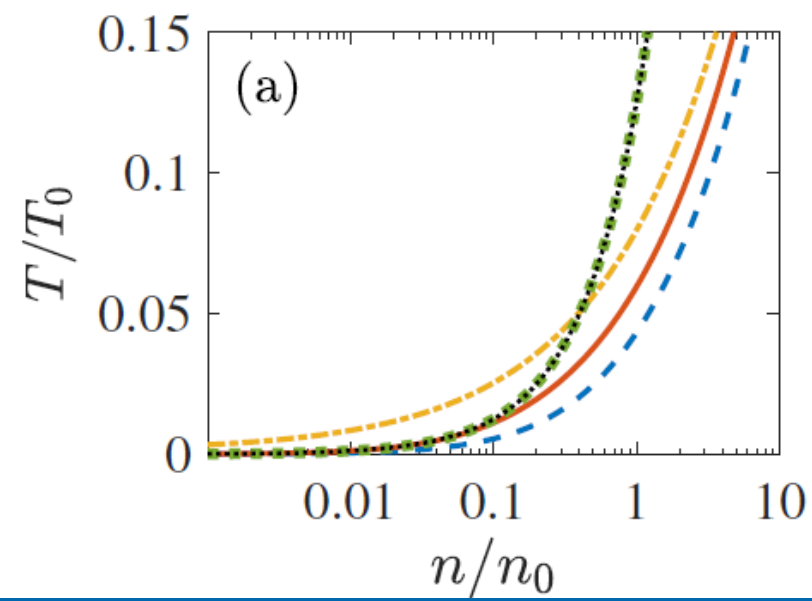
Bose Liquid

0.125

S. S. Botelho and C. A. R. Sá de Melo

PRL 96, 040404 (2006)

Conclusions in Pictures



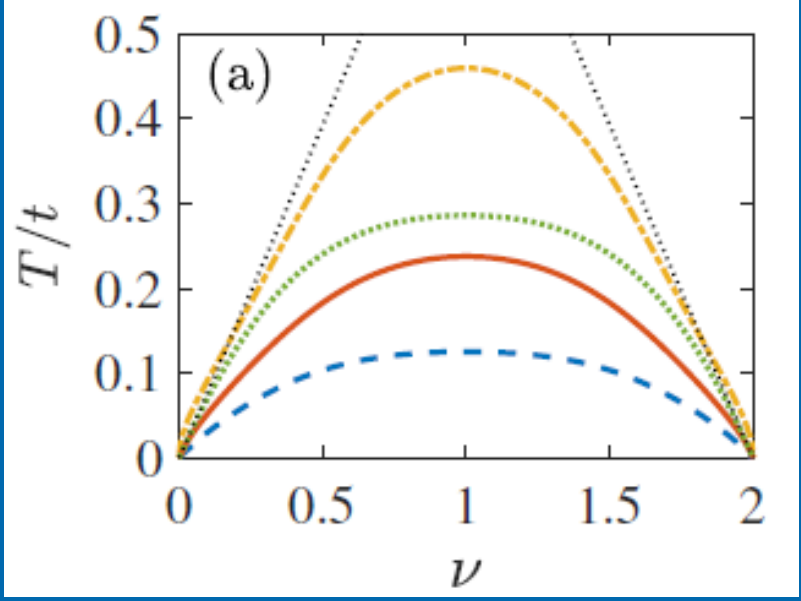


Table 1. List of temperatures obtained by different methods.

T_c	Description $T_c^{\mathrm{sup}} \leq T_c^{\theta} \leq T_c^{\mathrm{up0}} \leq T_c^{\mathrm{up1}}$
T_c^{sup}	Critical temperature supremum
T_c^{θ}	Phase-fluctuation critical temperature
T_c^{up0}	Upper bound set by superfluid density $\rho_s(T=0)$
T_c^{up1}	Upper bound set by kinetic energy
T_c^{mf}	Mean-field critical temperature
$T_c^{\theta\mathrm{mod}}$	Modulus and phase-fluctuation T_c

End of Lecture II