Nonequilibrium quantum phase transitions in atom-optomechanical systems

Milan Radonjić[1,2]

1. I. Institut für Theoretische Physik, Universität Hamburg, Germany

2. Institute of Physics Belgrade, University of Belgrade, Serbia

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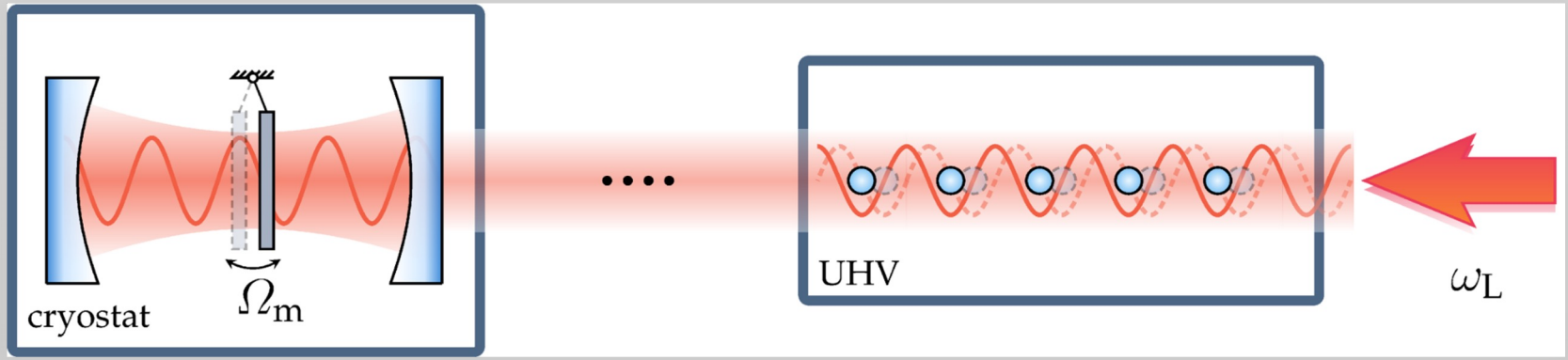




Part 1 outline

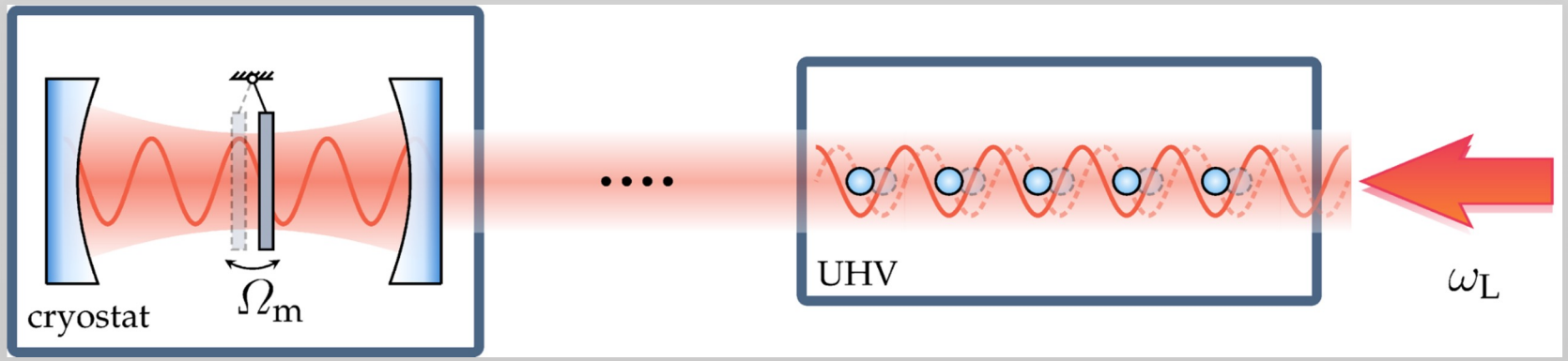
- Hybrid atom-optomechanical systems
- Distant motional coupling scheme
- Adiabatic light field elimination
- Quantum Langevin equations
- Mean-field approximation
- Cumulant expansion approach
- Nonequilibrium quantum phase transition

Hybrid atom-optomechanical systems



- low-finesse cavity which hosts a semitransparent nanomembrane
- light-mediated effective atom-membrane coupling
- N identical ultracold bosonic atoms held in an external trap
- pumped by an external laser drive
- motional/internal state coupling scheme

Distant motional coupling scheme

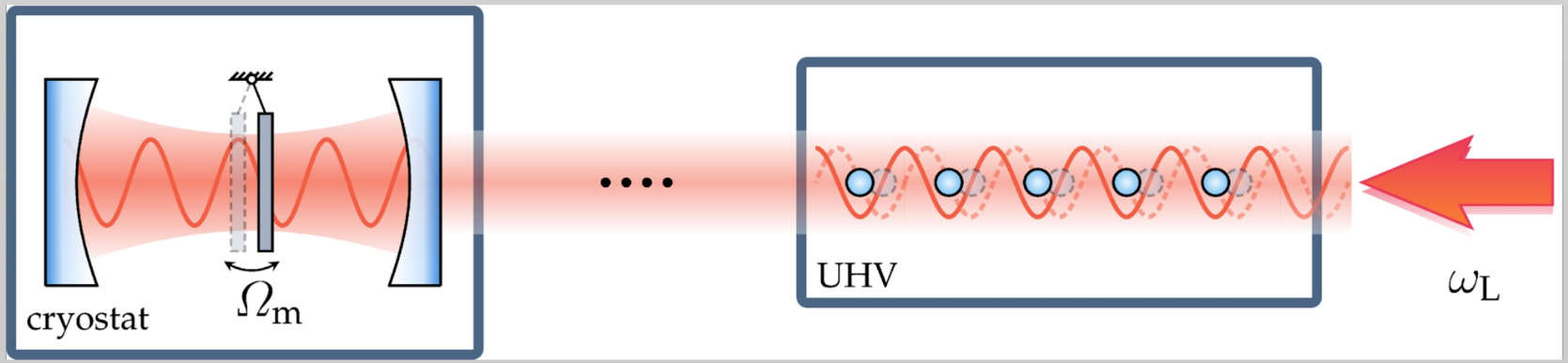


System Hamiltonian: $\hat{H} = \hat{H}_{\mathrm{m}} + \hat{H}_{\mathrm{a}} + \hat{H}_{\mathrm{l}} + \hat{H}_{\mathrm{m-l}} + \hat{H}_{\mathrm{a-l}}$

$$\hat{H}_{\mathrm{m}} = \Omega_{\mathrm{m}} \hat{a}^{\dagger} \hat{a}, \quad \hat{H}_{\mathrm{l}} = \int_{\omega_{\mathrm{L}} - \Delta\omega}^{\omega_{\mathrm{L}} + \Delta\omega} d\omega \, \omega \hat{b}_{\omega}^{\dagger} \hat{b}_{\omega}$$

$$\hat{H}_{\mathrm{a}} = -\frac{1}{2m} \int dz \, \hat{\Psi}^{\dagger}(z) \partial_z^2 \hat{\Psi}(z) + \frac{g}{2} \int dz \, \hat{\Psi}^{\dagger}(z) \hat{\Psi}^{\dagger}(z) \hat{\Psi}(z) \hat{\Psi}(z)$$

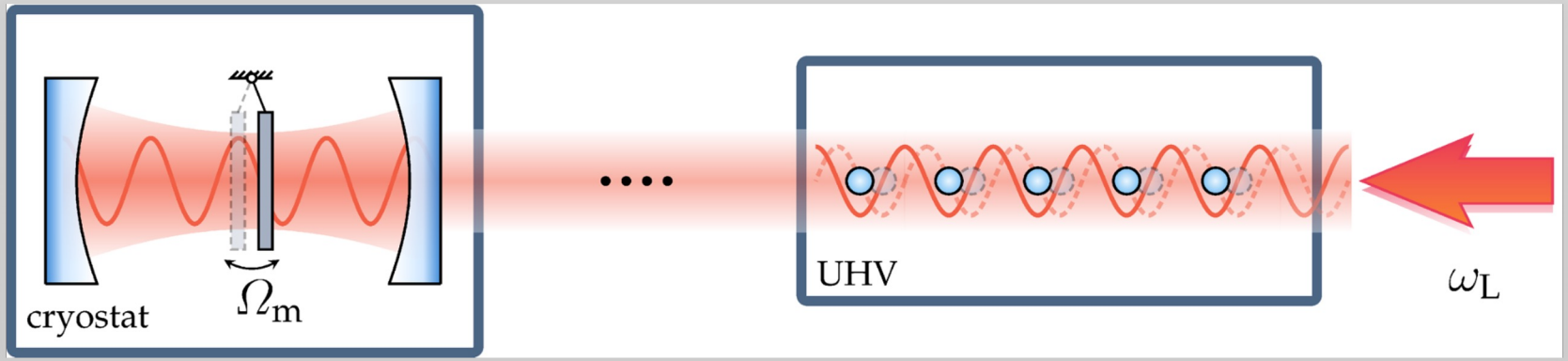
Distant motional coupling scheme



$$\text{Light field: } \hat{E}^{+}(z) = i \int d\omega\, \mathcal{E}_\omega u_\omega(z) \hat{b}_\omega, \quad \mathcal{E}_\omega = \sqrt{\frac{\omega}{\pi \mathcal{A}}}$$

$$u_\omega(z) = \begin{cases} |T_\omega| \sin(\omega z), & \text{if } 0 < z < \ell \\ |T_\omega A_\omega^{-1}| \sin(\omega z + \phi'_\omega), & \text{if } \ell + d_{\mathrm{m}} < z < L \\ \sin(\omega z + \phi_\omega), & \text{if } L + d < z \end{cases}$$

Distant motional coupling scheme



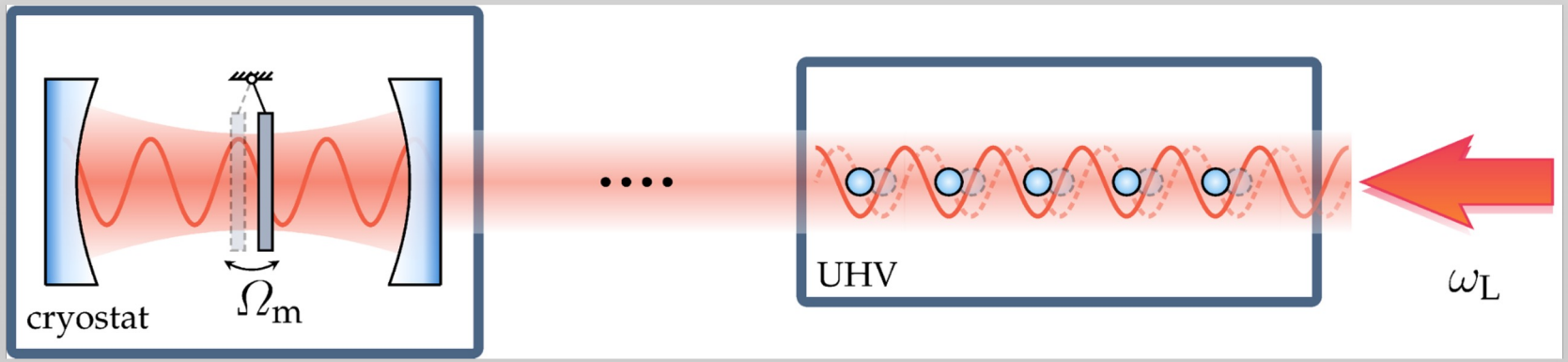
Membrane-light and atom-light coupling:

$$\hat{P}_{\mathrm{l}} = \frac{n_{\mathrm{m}}^2 - 1}{2}\left[\hat{E}^{-}(\ell + d_{\mathrm{m}})\hat{E}^{+}(\ell + d_{\mathrm{m}}) - \hat{E}^{-}(\ell)\hat{E}^{+}(\ell)\right]$$

$$\hat{H}_{\mathrm{m-l}} = \hat{P}_{\mathrm{l}}\mathcal{A}\hat{z}_{\mathrm{m}} = \hat{P}_{\mathrm{l}}\mathcal{A}\frac{\hat{a} + \hat{a}^{\dagger}}{\sqrt{2M\Omega_{\mathrm{m}}}}$$

$$\hat{H}_{\mathrm{a-l}} = \frac{\mu_{\mathrm{a}}^2}{\Delta}\int dz\, \hat{E}^{-}(z)\hat{E}^{+}(z)\hat{\Psi}^{\dagger}(z)\hat{\Psi}(z)$$

Linearization of the Hamiltonian



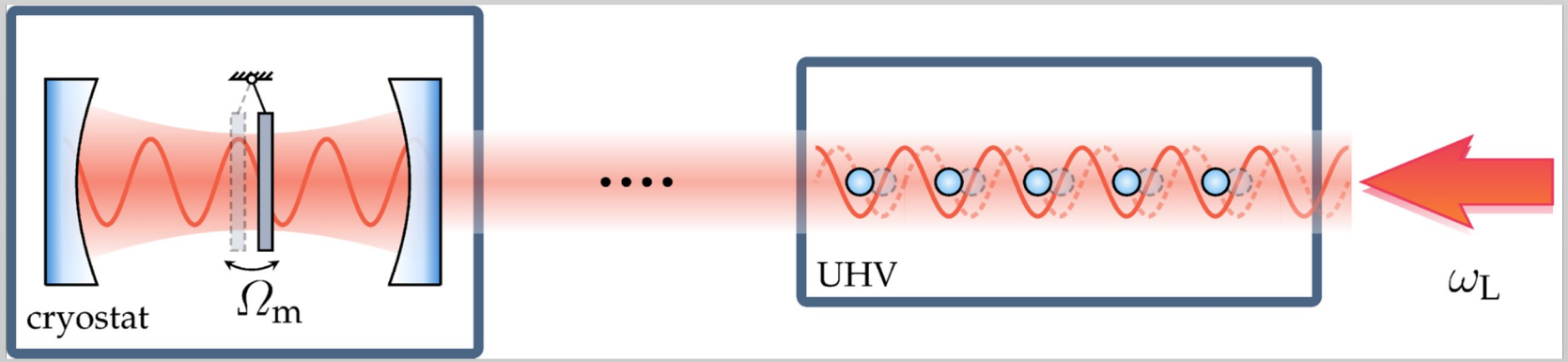
Linearization in the light field operators $\hat{b}_\omega, \hat{b}_\omega^\dagger$:

$\hat{b}_\omega \to \hat{b}_\omega + \alpha_{\mathrm{L}}\delta(\omega - \omega_{\mathrm{L}})e^{-i\omega_{\mathrm{L}}t}$ to include laser drive

$$|T_\omega|^2\Big|_{\omega \backsimeq \omega_{\mathrm{L}}} \approx \frac{2\mathcal{F}}{\pi}\frac{\kappa^2}{(\omega-\omega_{\mathrm{L}})^2+\kappa^2} \approx \frac{2\mathcal{F}}{\pi}, \quad \mathcal{F} = \frac{\pi}{2\kappa L}$$

$$u_\omega(z) \approx u_{\omega_{\mathrm{L}}}(z), \quad |u_{\omega_{\mathrm{L}}}(\ell + d_{\mathrm{m}})|^2 - |u_{\omega_{\mathrm{L}}}(\ell)|^2 \approx d_{\mathrm{m}}\partial_\ell |u_{\omega_{\mathrm{L}}}(\ell)|^2$$

Linearization of the Hamiltonian



Linearized membrane-light Hamiltonian:

$$\hat{H}^{\rm lin}_{\rm m-l} = \lambda_{\rm m}(\hat{a}+\hat{a}^\dagger)\int \frac{d\omega}{\sqrt{2\pi}}\left(\hat{b}_\omega e^{i\omega_L t}+\hat{b}^\dagger_\omega e^{-i\omega_L t}\right)$$

$$\lambda_{\rm m} = \frac{\alpha_L |r_{\rm m}|\omega_L}{\sqrt{\Omega_{\rm m} M}}\sin(2\omega_L \ell)\frac{2\mathcal{F}}{\pi^{3/2}}, \qquad |r_{\rm m}| = \frac{(n_{\rm m}^2-1)\omega_L d_{\rm m}}{2}$$

See: New J. Phys. **21**, 113037 (2019)

Linearization of the Hamiltonian

Linearized atom-light Hamiltonian:

$$\hat{H}^{\mathrm{lin}}_{\mathrm{a-l}} = \lambda_{\mathrm{a}} \int \frac{d\omega}{\sqrt{2\pi}} \left(\hat{b}_\omega e^{i\omega_L t} + \hat{b}^\dagger_\omega e^{-i\omega_L t}\right) \times \int dz\, \hat{\Psi}^\dagger(z) \sin(\omega_L z) \sin(\omega z + \Delta\phi_\omega) \hat{\Psi}(z)$$

$$\lambda_{\mathrm{a}} = \sqrt{2\pi}\, \alpha_{\mathrm{L}} \mathcal{E}^2_{\omega_L} \frac{\mu^2_{\mathrm{a}}}{\Delta}, \quad \Delta\phi_\omega \approx \phi_\omega - \phi_{\omega_L}$$

Optical lattice potential:

$$V(z) = V \sin^2(\omega_L z + \phi_{\omega_L}), \quad V = \alpha^2_{\mathrm{L}} \mathcal{E}^2_{\omega_L} \frac{\mu^2_{\mathrm{a}}}{\Delta}$$

Linearization of the Hamiltonian

Full linearized motional coupling scheme Hamiltonian:

$$\hat{H}^{\mathrm{lin}}_{\mathrm{mcs}} = \Omega_{\mathrm{m}}\hat{a}^{\dagger}\hat{a} + \int dz\, \hat{\Psi}^{\dagger}(z)\Big[-\frac{\partial_z^2}{2m} + V\sin^2(\omega_{\mathrm{L}} z) + \frac{g}{2}\hat{\Psi}^{\dagger}(z)\hat{\Psi}(z)\Big]\hat{\Psi}(z)$$
$$+ \int d\omega\, \Delta_{\omega}\hat{b}^{\dagger}_{\omega}\hat{b}_{\omega} + \lambda_{\mathrm{m}}(\hat{a}+\hat{a}^{\dagger})\int \frac{d\omega}{\sqrt{2\pi}}\left(\hat{b}_{\omega}+\hat{b}^{\dagger}_{\omega}\right)$$
$$+ \lambda_{\mathrm{a}}\int \frac{d\omega}{\sqrt{2\pi}}\left(\hat{b}_{\omega}+\hat{b}^{\dagger}_{\omega}\right)\int dz\, \hat{\Psi}^{\dagger}(z)\sin(\omega_L z)\sin(\omega z + \Delta\phi_{\omega})\hat{\Psi}(z)$$

Detuning from laser drive frequency: $\Delta_{\omega} = \omega - \omega_{\mathrm{L}}$

Light field modes will be adiabatically eliminated

Hierarchy of time scales $\Delta \gg \Delta\omega \gg \tau^{-1}, \kappa \gg \Omega_{\mathrm{m}}, \Omega_{\mathrm{a}}$

Adiabatic light field elimination

Go to interaction picture with respect to free evolution of photon field

$\hat{b}_\omega(t)_I = \hat{b}_\omega e^{-i\Delta_\omega t}$

Formal solution of the Schrödinger equation

$$|\psi(t)\rangle = \mathcal{T}\exp\left\{-i\int_{t_0}^{t} ds\, \hat{H}^{\mathrm{lin}}_{\mathrm{msc}}(s)_I\right\}|\psi(t_0)\rangle$$

We expand the above for small time steps δt

$$|\psi(t_0+\delta t)\rangle \approx \left\{1 - i\int_{t_0}^{t_0+\delta t} dt\, \hat{H}^{\mathrm{lin}}_{\mathrm{mcs}}(t)_I - \int_{t_0}^{t_0+\delta t} dt \int_{t_0}^{t} ds\, \hat{H}^{\mathrm{lin}}_{\mathrm{mcs}}(t)_I \hat{H}^{\mathrm{lin}}_{\mathrm{mcs}}(s)_I\right\}|\psi(t_0)\rangle$$

Initial state: $|\psi(t_0)\rangle = |\psi(t_0)\rangle_{\mathrm{a-m}} \otimes |\mathrm{vac}\rangle_{\mathrm{l}}$

Adiabatic light field elimination

Terms linear in δt from the single-integral term:

$$-i\left\{\hat{H}_{\mathrm{mcs},0}\delta t+\lambda_{\mathrm{m}}(\hat{a}+\hat{a}^{\dagger})\delta\hat{B}^{\dagger}(t_0)+\lambda_{\mathrm{a}}\int dz\,\hat{\Psi}^{\dagger}(z)\hat{\Psi}(z)\delta\hat{C}^{\dagger}(t_0,z)\right\}|\psi(t_0)\rangle$$

$$\hat{H}_{\mathrm{mcs},0}=\Omega_{\mathrm{m}}\hat{a}^{\dagger}\hat{a}+\int dz\,\hat{\Psi}^{\dagger}(z)\left[-\frac{\partial_z^2}{2m}+V\sin^2(\omega_{\mathrm{L}}z)+\frac{g}{2}\hat{\Psi}^{\dagger}(z)\hat{\Psi}(z)\right]\hat{\Psi}(z)$$

Noise-increment operators:

$$\delta\hat{B}(t)=\hat{B}(t+\delta t)-\hat{B}(t),\quad \delta\hat{C}(t,z)=\hat{C}(t+\delta t,z)-\hat{C}(t,z)$$

$$\hat{B}(t)=\int_{t_0}^{t}ds\int\frac{d\omega}{\sqrt{2\pi}}\hat{b}_{\omega}(s)_I$$

$$\hat{C}(t,z)=\int_{t_0}^{t}ds\int\frac{d\omega}{\sqrt{2\pi}}\sin(\omega_{\mathrm{L}}z)\sin(\omega z+\Delta\phi_{\omega})\hat{b}_{\omega}(s)_I$$

Adiabatic light field elimination

Terms linear in δt from the double-integral term:

$$-i\Big\{-\lambda(\hat{a}+\hat{a}^\dagger)\int dz\,\hat{\Psi}^\dagger(z)\sin(2\omega_{\mathrm{L}}z)\hat{\Psi}(z)$$

$$-2i\lambda(\hat{a}+\hat{a}^\dagger)\int dz\,\hat{\Psi}^\dagger(z)\sin^2(\omega_{\mathrm{L}}z)\hat{\Psi}(z)-i\frac{\lambda_{\mathrm{m}}^2}{2}(\hat{a}+\hat{a}^\dagger)^2$$

$$+\frac{\lambda_{\mathrm{a}}^2}{4}\int dz\int dz'\,\hat{\Psi}^\dagger(z)\hat{\Psi}(z)\mathcal{G}(z,z')\hat{\Psi}^\dagger(z')\hat{\Psi}(z')\Big\}\delta t\,|\psi(t_0)\rangle$$

Effective atom-membrane coupling strength $\lambda=\lambda_{\mathrm{m}}\lambda_{\mathrm{a}}/2$

$$\mathcal{G}(z,z')=\sin(\omega_{\mathrm{L}}z)\sin(\omega_{\mathrm{L}}z')\big[\sin(\omega_{\mathrm{L}}|z-z'|)$$
$$-\sin(\omega_{\mathrm{L}}[z+z'])-2i\sin(\omega_{\mathrm{L}}z)\sin(\omega_{\mathrm{L}}z')\big]$$

Stochastic Schrödinger equation

In limit $\delta t \to 0$, quantum stochastic Schrödinger equation is found:

$$d|\psi(t)\rangle = \left\{ -i\hat{H}^{\text{eff}}_{\text{mcs}} - \frac{1}{2}\left[\lambda_{\text{m}}(\hat{a}+\hat{a}^{\dagger}) + \lambda_{\text{a}}\int dz\,\hat{\Psi}^{\dagger}(z)\sin^2(\omega_{\text{L}}z)\hat{\Psi}(z)\right]^2 \right\}|\psi(t)\rangle\,dt$$
$$- i\left\{\lambda_{\text{m}}(\hat{a}+\hat{a}^{\dagger})d\hat{B}^{\dagger}(t) + \lambda_{\text{a}}\int dz\,\hat{\Psi}^{\dagger}(z)\hat{\Psi}(z)d\hat{C}^{\dagger}(t,z)\right\}|\psi(t)\rangle$$

Effective atom-membrane Hamiltonian:

$$\hat{H}^{\text{eff}}_{\text{mcs}} = \Omega_{\text{m}}\hat{a}^{\dagger}\hat{a} + \int dz\,\hat{\Psi}^{\dagger}(z)\left[-\frac{\partial_z^2}{2m} + V\sin^2(\omega_{\text{L}}z) + \frac{g}{2}\hat{\Psi}^{\dagger}(z)\hat{\Psi}(z)\right]\hat{\Psi}(z)$$
$$- \lambda(\hat{a}+\hat{a}^{\dagger})\int dz\,\hat{\Psi}^{\dagger}(z)\sin(2\omega_{\text{L}}z)\hat{\Psi}(z)$$
$$+ \frac{\lambda_{\text{a}}^2}{4}\int dz\int dz'\,\hat{\Psi}^{\dagger}(z)\hat{\Psi}(z)\text{Re}[\mathcal{G}(z,z')]\hat{\Psi}^{\dagger}(z')\hat{\Psi}(z')$$

Stochastic Schrödinger equation

In limit $\delta t \to 0$, quantum stochastic Schrödinger equation is found:

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$$- i\left\{ \lambda_{\text{m}}(\hat{a}+\hat{a}^{\dagger})d\hat{B}^{\dagger}(t) + \lambda_{\text{a}}\int dz\, \hat{\Psi}(z)\hat{\Psi}(z)d\hat{C}^{\dagger}(t,z) \right\}|\psi(t)\rangle$$

Stochastic differential increments:

$$d|\psi(t)\rangle = \lim_{\delta t\to 0}\left[|\psi(t+\delta t)\rangle - |\psi(t)\rangle\right]$$
$$d\hat{B}(t) = \lim_{\delta t\to 0}\left[B(t+\delta t) - B(t)\right]$$
$$d\hat{C}(t,z) = \lim_{\delta t\to 0}\left[C(t+\delta t,z) - C(t,z)\right]$$

Stochastic Schrödinger equation

In limit $\delta t \to 0$, quantum stochastic Schrödinger equation is found:

$$d|\psi(t)\rangle = \left\{ -i\hat{H}^{\text{eff}}_{\text{mcs}} - \frac{1}{2}\left[\lambda_{\text{m}}(\hat{a}+\hat{a}^{\dagger}) + \lambda_{\text{a}}\int dz\, \hat{\Psi}^{\dagger}(z)\sin^2(\omega_{\text{L}}z)\hat{\Psi}(z)\right]^2 \right\}|\psi(t)\rangle dt$$
$$- i\left\{ \lambda_{\text{m}}(\hat{a}+\hat{a}^{\dagger})d\hat{B}^{\dagger}(t) + \lambda_{\text{a}}\int dz\, \hat{\Psi}(z)\hat{\Psi}(z)d\hat{C}^{\dagger}(t,z) \right\}|\psi(t)\rangle$$

Differential noise operators follow Itô rules:

$$d\hat{B}(t)d\hat{B}^{\dagger}(t) = dt$$
$$d\hat{C}(t,z)d\hat{C}^{\dagger}(t,z') = \sin^2(\omega_{\text{L}}z)\sin^2(\omega_{\text{L}}z')dt$$
$$d\hat{B}(t)d\hat{C}^{\dagger}(t,z) = d\hat{C}(t,z)d\hat{B}^{\dagger}(t) = \sin^2(\omega_{\text{L}}z)dt$$

Stochastic Schrödinger equation

In limit $\delta t \to 0$, quantum stochastic Schrödinger equation is found:

$$d|\psi(t)\rangle = \left\{ -i\hat{H}^{\text{eff}}_{\text{mcs}} - \frac{1}{2}\left[\lambda_{\text{m}}(\hat{a}+\hat{a}^{\dagger}) + \lambda_{\text{a}}\int dz\, \hat{\Psi}^{\dagger}(z)\sin^2(\omega_{\text{L}}z)\hat{\Psi}(z)\right]^2 \right\}|\psi(t)\rangle dt$$
$$- i\left\{ \lambda_{\text{m}}(\hat{a}+\hat{a}^{\dagger})d\hat{B}^{\dagger}(t) + \lambda_{\text{a}}\int dz\, \hat{\Psi}(z)\hat{\Psi}(z)d\hat{C}^{\dagger}(t,z)\right\}|\psi(t)\rangle$$

There exists unitary stochastic time evolution operator for QSSE:

$$\hat{U}(t,t_0) = \mathcal{T}\exp\left\{ -i\int_{t_0}^{t}\left(\hat{H}^{\text{eff}}_{\text{mcs}}ds + \lambda_{\text{m}}(\hat{a}+\hat{a}^{\dagger})\left[d\hat{B}(s) + d\hat{B}^{\dagger}(s)\right] \right.\right.$$
$$\left.\left. + \lambda_{\text{a}}\int dz\, \hat{\Psi}^{\dagger}(z)\hat{\Psi}(z)\left[d\hat{C}(s,z) + d\hat{C}^{\dagger}(s,z)\right]\right)\right\}$$

Quantum Langevin equations

Stochastic evolution is governed by $\hat{U}(t, t_0)$:

$$|\psi(t)\rangle = \hat{U}(t, t_0)|\psi(t_0)\rangle, \quad \hat{O}(t) = \hat{U}^\dagger(t, t_0)\hat{O}(t_0)\hat{U}(t, t_0)$$

Quantum Langevin equations:

$$d\hat{a} = -i[\hat{a}, \hat{H}^{\text{eff}}_{\text{mcs}}]dt - i\lambda_{\text{m}}(d\hat{B} + d\hat{B}^\dagger)$$
$$d\hat{\Psi}(z) = -i[\hat{\Psi}(z), \hat{H}^{\text{eff}}_{\text{mcs}}]dt - i\lambda_{\text{a}} \sin^2(\omega_{\text{L}} z)\hat{\Psi}(z)(d\hat{B} + d\hat{B}^\dagger)$$

with

$$d\hat{C}(t, z) \mapsto \sin^2(\omega_{\text{L}} z)d\hat{B}(t),$$

which satisfies the same Itô rules

Quantum Langevin equations

We interpret $d\hat{B} + d\hat{B}^\dagger$ as increments of white noise operators $\hat{\xi}_\mathrm{l}$:

$$i\partial_t \hat{a} = [\hat{a}, \hat{H}^\mathrm{eff}_\mathrm{mcs}] - \lambda_\mathrm{m}\hat{\xi}_\mathrm{l}$$
$$i\partial_t \hat{\Psi}(z) = [\hat{\Psi}(z), \hat{H}^\mathrm{eff}_\mathrm{mcs}] - \lambda_\mathrm{a}\sin^2(\omega_\mathrm{L} z)\hat{\Psi}(z)\hat{\xi}_\mathrm{l}$$

Including thermal damping of membrane gives final QLEs:

$$i\partial_t \hat{a} = (\Omega_\mathrm{m} - i\Gamma_\mathrm{m})\hat{a} - \lambda \int dz\, \hat{\Psi}^\dagger(z)\sin(2z)\hat{\Psi}(z) - \lambda_\mathrm{m}\hat{\xi}_\mathrm{l} - \hat{\xi}_\mathrm{th}$$
$$i\partial_t \hat{\Psi}(z) = \big[-\omega_\mathrm{R}\partial_z^2 + V\sin^2(z) + g\hat{\Psi}^\dagger(z)\hat{\Psi}(z) - \lambda(\hat{a} + \hat{a}^\dagger)\sin(2z)\big]\hat{\Psi}(z)$$
$$- \lambda_\mathrm{a}\sin^2(z)\hat{\Psi}(z)\hat{\xi}_\mathrm{l}$$

where $\omega_\mathrm{R} = \omega_\mathrm{L}^2/2m$ and we rescaled space coordinate $z \mapsto z/\omega_\mathrm{L}$

Autocorrelation functions: $\langle \hat{\xi}_\mathrm{l}(t)\hat{\xi}_\mathrm{l}(s)\rangle = \delta(t-s)$, $\langle \hat{\xi}_\mathrm{th}(t)\hat{\xi}^\dagger_\mathrm{th}(s)\rangle = 2\Gamma_\mathrm{m}\delta(t-s)$

Mean-field approximation

Atoms are condensed and membrane is in coherent state, so that $\hat{\Psi}(z) \approx \sqrt{N}\psi(z)$ and $\hat{a} \approx \sqrt{N}\alpha$:

$$i\partial_t \alpha = (\Omega_{\rm m} - i\Gamma_{\rm m})\alpha - \sqrt{N}\lambda \int dz\, \sin(2z)|\psi(z)|^2$$

$$i\partial_t \psi(z) = \big[-\omega_{\rm R}\partial_z^2 + V\sin^2(z) + Ng|\psi(z)|^2 - \sqrt{N}\lambda(\alpha + \alpha^*)\sin(2z)\big]\psi(z)$$

Steady state is described by ansatz $\psi(t,z) = e^{-i\mu t}\psi_0(z)$ and $\alpha(t) = \alpha_0$

$$\alpha_0 = \frac{\sqrt{N}\lambda}{\Omega_{\rm m} - i\Gamma_{\rm m}} \int dz\, \sin(2z)|\psi_0(z)|^2$$

$$\mu\psi_0(z) = \big[-\omega_{\rm R}\partial_z^2 + V\sin^2(z) + Ng|\psi_0(z)|^2 - \sqrt{N}\lambda(\alpha_0 + \alpha_0^*)\sin(2z)\big]\psi_0(z)$$

Dynamical competition between $V\sin^2(z)$ and $\sqrt{N}\lambda(\alpha_0 + \alpha_0^*)\sin(2z)$ leads to nonequilibrium phase transitions

Cumulant expansion approach

In deep optical lattice, when $V \gg \omega_{\mathrm{R}}$, condensate is well described by sum of Gaussians residing in individual lattice wells

$$\psi(t,z) = \frac{1}{\pi^{1/4}\sqrt{\sigma(t)}} \exp\left(-\frac{[z-\zeta(t)]^2}{2\sigma(t)^2} + i\kappa(t)z + i\eta(t)z^2 \right)$$

$\int dz\, \psi^*(t,z)[z-\zeta(t)] \times$ EOM for $\psi(t)$ produces:

$$i\frac{\dot{\zeta}}{2} - \frac{\sigma^2}{2}(\dot{\kappa} + 2\dot{\eta}\zeta) = \omega_{\mathrm{R}}(\kappa + 2\eta\zeta)(i + 2\eta\sigma^2) + \left[\frac{V}{2}\sin(2\zeta) - \sqrt{N}\lambda(\alpha + \alpha^*)\cos(2\zeta)\right]\sigma^2 e^{-\sigma^2}$$

Cumulant expansion approach

2nd cumulant EOM by multiplying with $\psi^*(t,z)\{[z-\zeta(t)]^2-\sigma(t)^2/2\}$ and integrating over z to get

$$i\frac{\sigma\dot{\sigma}}{2}-\frac{\dot{\eta}\sigma^4}{4}=\omega_{\mathrm{R}}\left[2\eta^2\sigma^4-\frac{1}{2}+2i\eta\sigma^2\right]-\frac{Ng\sigma}{4\sqrt{2\pi}}$$
$$+\left[\frac{V}{2}\cos(2\zeta)+\sqrt{N}\lambda(\alpha+\alpha^*)\sin(2\zeta)\right]\sigma^4 e^{-\sigma^2}$$

Imaginary parts of two cumulant EOMs yield

$$\dot{\zeta}=2\omega_{\mathrm{R}}(\kappa+2\eta\zeta),\quad \dot{\sigma}=4\omega_{\mathrm{R}}\eta\sigma$$

so that η and κ can be eliminated

Cumulant expansion approach

After elimination, real parts yield equations of motion:

$$\dot{\alpha} = -i\partial_{\alpha^*} E - \Gamma_{\mathrm{m}}\alpha, \quad (2\omega_{\mathrm{R}})^{-1}\ddot{\zeta} = -\partial_{\zeta} E, \quad (4\omega_{\mathrm{R}})^{-1}\ddot{\sigma} = -\partial_{\sigma} E$$

Effective potential energy:

$$E[\alpha, \mathcal{S}, \sigma] = \Omega_{\mathrm{m}}|\alpha|^2 + \frac{\omega_{\mathrm{R}}}{2\sigma^2} + \frac{Ng}{\sqrt{8\pi}\sigma} - \left[\frac{V}{2}\sqrt{1-\mathcal{S}^2} - \sqrt{N}\lambda(\alpha+\alpha^*)\mathcal{S}\right] e^{-\sigma^2}$$

with order parameter $\mathcal{S} = \sin(2\zeta)$

See: Phys. Rev. Lett. **120**, 063605 (2018)

Nonequilibrium quantum phase transition

Using membrane displacement

$$\alpha_0 = \frac{\sqrt{N}\lambda \mathcal{S}_0}{\Omega_{\mathrm{m}} - i\Gamma_{\mathrm{m}}} e^{-\sigma_0^2},$$

steady state is found from:

$$\sqrt{1-\mathcal{S}_0^2}\left[\omega_{\mathrm{R}} + \frac{Ng}{\sqrt{8\pi}}\sigma_0\right] = V\sigma_0^4 e^{-\sigma_0^2}$$

$$\mathcal{S}_0\left[N\lambda^2\sqrt{1-\mathcal{S}_0^2} - N\lambda_V^2 e^{\sigma_0^2}\right] = 0$$

where $\sqrt{N}\lambda_V = \sqrt{\dfrac{V(\Omega_{\mathrm{m}}^2 + \Gamma_{\mathrm{m}}^2)}{4\Omega_{\mathrm{m}}}}$

Nonequilibrium quantum phase transition

2nd equation has 3 possible solutions:

$$\mathcal{S}_0 = 0, \quad \text{for } \lambda \lesssim \lambda_V$$

$$\mathcal{S}_0 = \pm\sqrt{1 - (\lambda_V/\lambda)^4 e^{2\sigma_0^2}}, \quad \text{for } \lambda \gtrsim \lambda_V$$

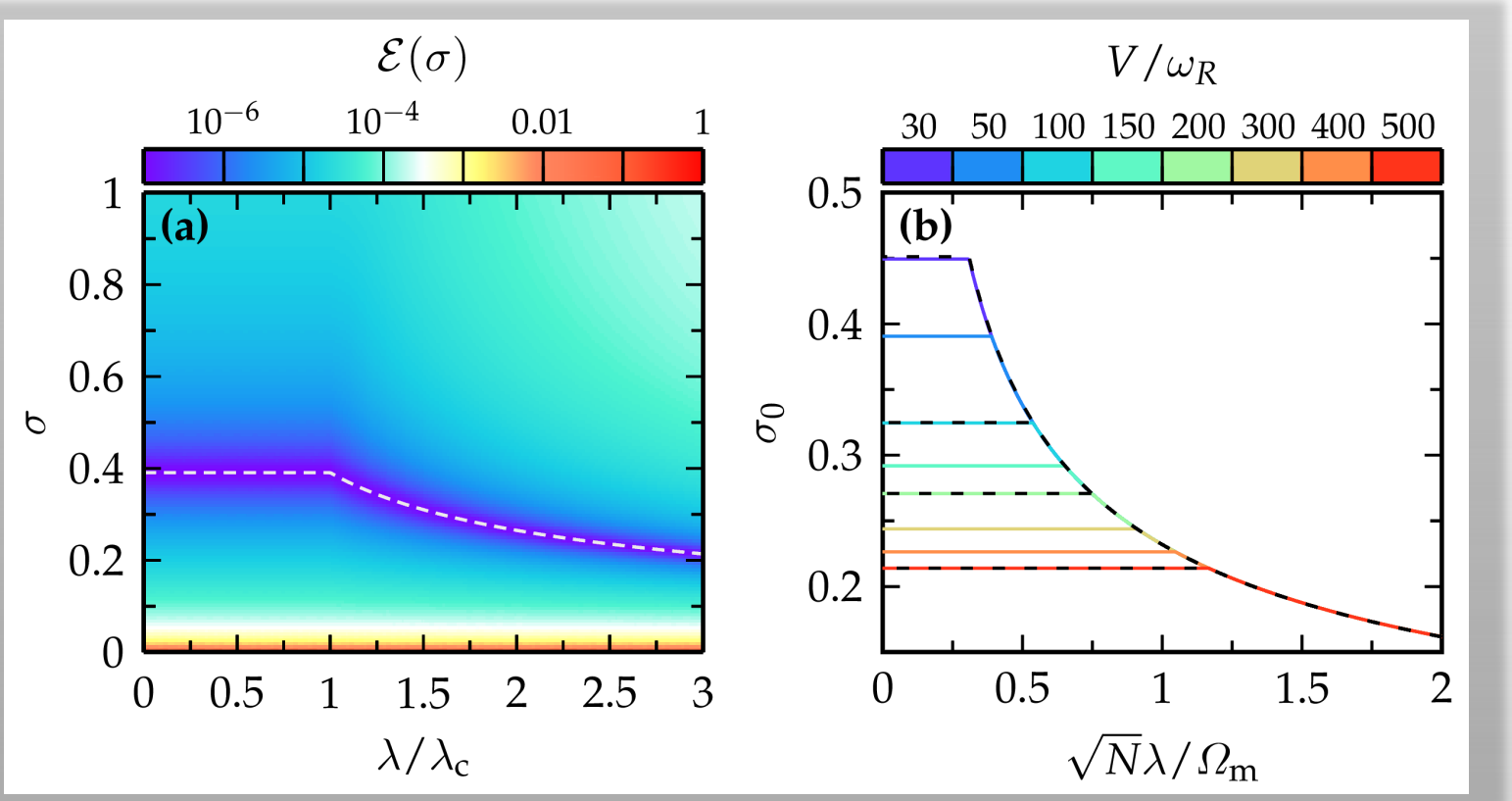
where σ_0 is to be found from 1st equation

Consider normalized potential energy:

$$\mathcal{E}(\sigma) = \frac{E(\sigma) - E(\sigma_0)}{\max_{\bar{\sigma}}\left[E(\bar{\sigma}) - E(\sigma_0)\right]}, \quad E(\sigma) = E[\alpha_0(\sigma), \mathcal{S}_0(\sigma), \sigma]$$

exhibiting minimum $\mathcal{E}(\sigma_0) = 0$

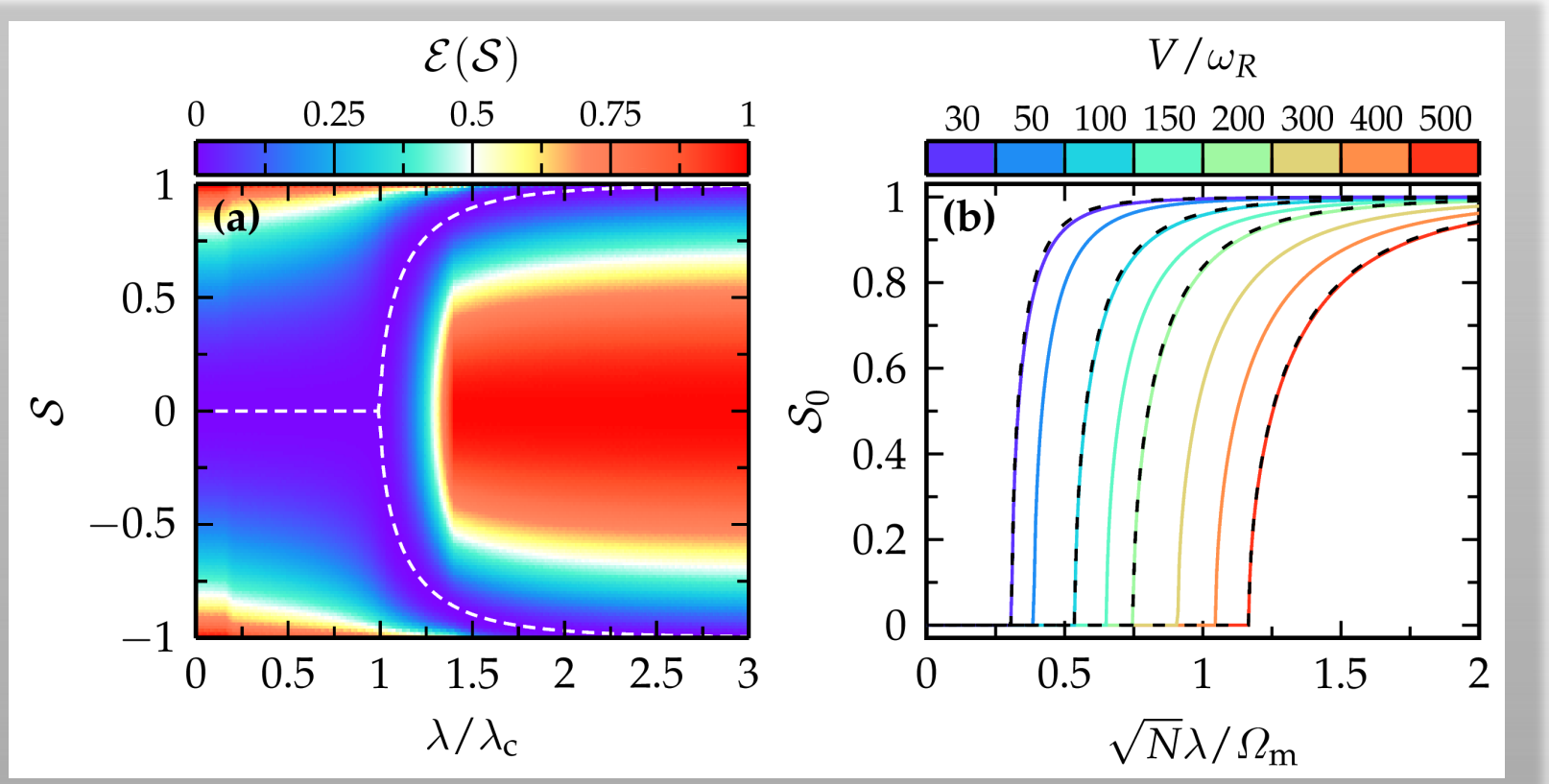
Nonequilibrium quantum phase transition



$$\omega_R + \frac{Ng}{2\sqrt{\pi}}\sqrt{\log\frac{\lambda_c}{\lambda_V}} = 4V\left(\frac{\lambda_V}{\lambda_c}\log\frac{\lambda_c}{\lambda_V}\right)^2$$

(a) Non-interacting atoms with $V = 50\omega_R$. For a fixed λ, $\mathcal{E}(\sigma)$ exhibits a global minimum which defines the steady state value σ_0 as indicated by the dashed curve. NQPT occurs at $\lambda = \lambda_c$. (b) Stationary condensate width σ_0 as a function of the coupling strength for various potential depths V, as indicated by the color bar.

Nonequilibrium quantum phase transition



$$\omega_R + \frac{Ng}{2\sqrt{\pi}}\sqrt{\log\frac{\lambda_c}{\lambda_V}}$$

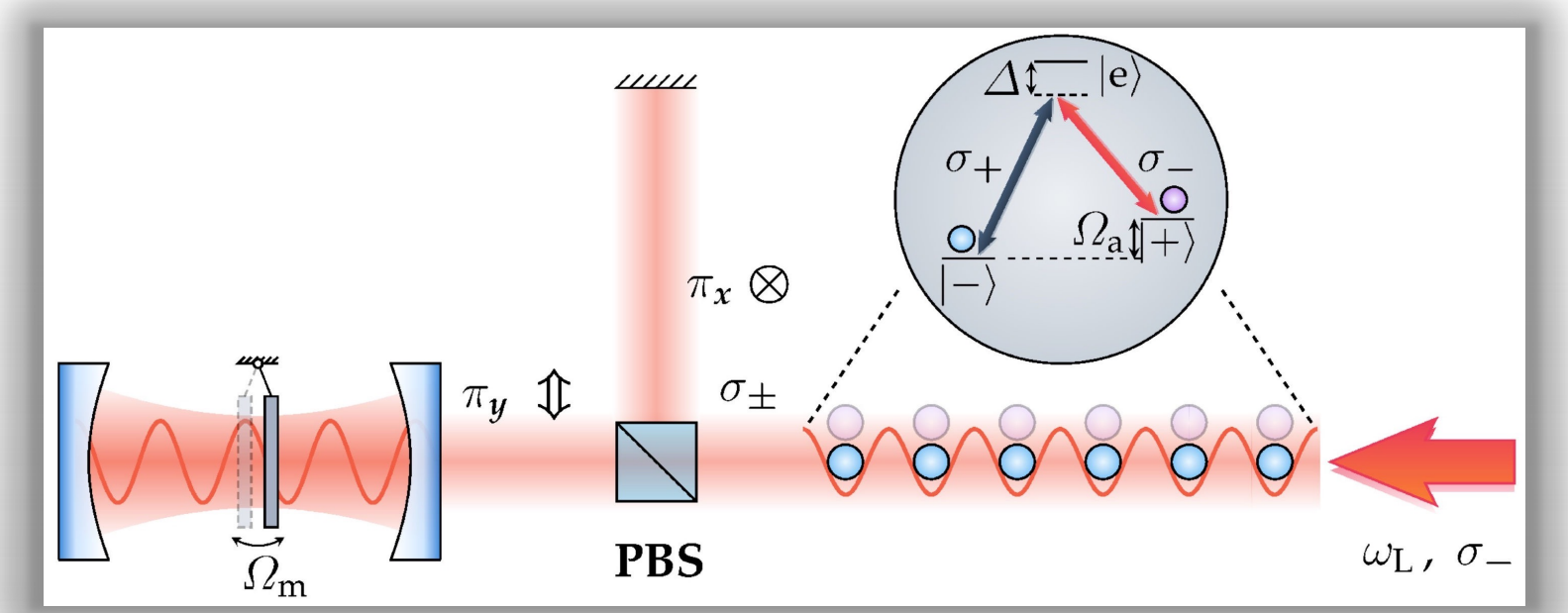
$$= 4V\left(\frac{\lambda_V}{\lambda_c}\log\frac{\lambda_c}{\lambda_V}\right)^2$$

(a) Non-interacting atoms with $V = 50\omega_R$. Below the critical coupling strength the potential surface exhibits a single minimum at $\mathcal{S} = 0$. At $\lambda = \lambda_c$, this minimum turns into a maximum and two new minima are formed at $\mathcal{S} \neq 0$. (b) Stationary value of the order parameter $\mathcal{S}_0$ for various potential depths V, as indicated by the color bar.

Part 2 outline

- Internal state coupling scheme
- Changing order of NQPT
- Cumulant expansion approach
- NQPTs of 1^{st} and 2^{nd} order
- Back to mean-field equations of motion
- Homogeneous / inhomogeneous ground steady states
- Further NQPTs

Internal state coupling scheme



- N identical ultracold bosonic atoms held in an external quasi-1D trap
- atomic states $\{|-\rangle, |+\rangle, |e\rangle\}$ constitute a Λ level scheme
- low-finesse cavity which hosts a semitransparent nanomembrane
- light-mediated effective atom-membrane coupling

Changing order of NQPT

Effective internal state coupling scheme Hamiltonian:

$$\hat{H}_{\mathrm{ics}}^{\mathrm{eff}} = \Omega_{\mathrm{m}}\hat{a}^{\dagger}\hat{a} + \sum_{\nu=\pm}\int dz\,\hat{\Psi}_{\nu}^{\dagger}(z)\Bigg[\nu\frac{\Omega_{\mathrm{a}}}{2} - \frac{\partial_z^2}{2m} - \frac{V}{2}\sin(2\omega_{\mathrm{L}}z)$$

$$+ \frac{1}{2}\sum_{\nu'=\pm} g_{\nu\nu'}\hat{\Psi}_{\nu'}^{\dagger}(z)\hat{\Psi}_{\nu'}(z)\Bigg]\hat{\Psi}_{\nu}(z) - (\hat{a}+\hat{a}^{\dagger})\int dz\,\sin(2\omega_{\mathrm{L}}z)$$

$$\times\left\{\frac{\lambda}{2}\left[\hat{\Psi}_{+}^{\dagger}(z)\hat{\Psi}_{-}(z) + \hat{\Psi}_{-}^{\dagger}(z)\hat{\Psi}_{+}(z)\right] + \lambda_{\mathrm{ex}}\hat{\Psi}_{+}^{\dagger}(z)\hat{\Psi}_{+}(z)\right\}$$

$$+ \frac{1}{4}\int dz\int dz'\,\mathrm{Re}[\mathcal{G}(z,z')]\hat{\Psi}_{+}^{\dagger}(z)\left[\lambda_{\mathrm{a}}^2\hat{\Psi}_{+}(z)\hat{\Psi}_{+}^{\dagger}(z') + \lambda_{\pm}^2\hat{\Psi}_{-}(z)\hat{\Psi}_{-}^{\dagger}(z')\right]\hat{\Psi}_{+}(z')$$

with $\lambda = \frac{\lambda_{\mathrm{m}}\lambda_{\pm}}{2}, \lambda_{\mathrm{ex}} = \frac{\lambda_{\mathrm{m}}\lambda_{\mathrm{a}}}{2}$ and $\lambda_{\pm} = \sqrt{2\pi}\,\alpha_{\mathrm{L}}\mathcal{E}_{\omega_{\mathrm{L}}}^2\frac{\mu_{+}\mu_{-}}{\Delta}, \lambda_{\mathrm{a}} = \sqrt{2\pi}\,\alpha_{\mathrm{L}}\mathcal{E}_{\omega_{\mathrm{L}}}^2\frac{\mu_{+}^2}{\Delta}$

Changing order of NQPT

Quantum stochastic Schrödinger equation follows similarly as before:

$$
\begin{aligned}
d|\psi(t)\rangle = \Bigg\{ & -i\hat{H}^{\mathrm{eff}}_{\mathrm{ics}} - \frac{1}{2}\Bigg[\lambda_{\mathrm{m}}(\hat{a}+\hat{a}^{\dagger}) + \int dz\, \sin^2(\omega_{\mathrm{L}} z)\big[\lambda_{\mathrm{a}}\hat{\Psi}^{\dagger}_{+}(z)\hat{\Psi}_{+}(z) \\
& + \lambda_{\pm}\hat{\Psi}^{\dagger}_{+}(z)\hat{\Psi}_{-}(z) + \lambda_{\pm}\hat{\Psi}^{\dagger}_{-}(z)\hat{\Psi}_{+}(z)\big]\Bigg]^2\Bigg\}|\psi(t)\rangle dt \\
& - i\Bigg\{\lambda_{\mathrm{m}}(\hat{a}+\hat{a}^{\dagger})d\hat{B}^{\dagger}(t) + \lambda_{\mathrm{a}}\int dz\, \hat{\Psi}^{\dagger}_{+}(z)\hat{\Psi}_{+}(z)d\hat{C}^{\dagger}_{-}(t,z) \\
& + \lambda_{\pm}\int dz\, \big[\hat{\Psi}^{\dagger}_{+}(z)\hat{\Psi}_{-}(z) + \hat{\Psi}^{\dagger}_{-}(z)\hat{\Psi}_{+}(z)\big]d\hat{C}^{\dagger}_{+}(t,z)\Bigg\}|\psi(t)\rangle
\end{aligned}
$$

where noises $d\hat{B}(t), d\hat{C}_{\pm}(t,z)$ satisfy appropriate Itô rules

Changing order of NQPT

Quantum Langevin equations also follow:

$$i\partial_t \hat{a} = [\hat{a}, \hat{H}^{\mathrm{eff}}_{\mathrm{ics}}] - \lambda_{\mathrm{m}}(\hat{\xi}_+ + \hat{\xi}_-)$$
$$i\partial_t \hat{\Psi}_-(z) = [\hat{\Psi}_-(z), \hat{H}^{\mathrm{eff}}_{\mathrm{ics}}] - \lambda_{\pm} \sin^2(\omega_{\mathrm{L}} z)\hat{\Psi}_+(z)\hat{\xi}_+$$
$$i\partial_t \hat{\Psi}_+(z) = [\hat{\Psi}_+(z), \hat{H}^{\mathrm{eff}}_{\mathrm{ics}}] - \lambda_{\pm} \sin^2(\omega_{\mathrm{L}} z)\hat{\Psi}_-(z)\hat{\xi}_+ - \lambda_{\mathrm{a}} \sin^2(\omega_{\mathrm{L}} z)\hat{\Psi}_+(z)\hat{\xi}_-$$

where noises satisfy $\langle \hat{\xi}_\nu(t)\hat{\xi}_{\nu'}(s)\rangle = \delta_{\nu\nu'}\delta(t-s)$

Mean-field approximation $\hat{\Psi}_\pm(z) \approx \sqrt{N}\psi_\pm(z)$ and $\hat{a} \approx \sqrt{N}\alpha$ gives:

$$i\partial_t \alpha = (\Omega_{\mathrm{m}} - i\Gamma_{\mathrm{m}})\alpha - \sqrt{N}\lambda_{\mathrm{ex}} \int dz \cos(2z)|\psi_+(z)|^2$$
$$- \frac{\sqrt{N}\lambda}{2} \int dz \cos(2z)\left[\psi_+^*(z)\psi_-(z) + \psi_+(z)\psi_-^*(z)\right]$$

Changing order of NQPT

Mean-field condensate wavefunctions evolve according to:

$$i\partial_t\psi_-(z) = \left[-\frac{\Omega_\mathrm{a}}{2} - \omega_\mathrm{R}\partial_z^2 - \frac{V}{2}\cos(2z) + N\sum_{\nu=\pm} g_{\nu-}|\psi_\nu(z)|^2 \right]\psi_-(z)$$

$$- \frac{\sqrt{N}\lambda}{2}(\alpha+\alpha^*)\cos(2z)\psi_+(z)$$

$$i\partial_t\psi_+(z) = \left[\frac{\Omega_\mathrm{a}}{2} - \omega_\mathrm{R}\partial_z^2 - \frac{V}{2}\cos(2z) + N\sum_{\nu=\pm} g_{\nu+}|\psi_\nu(z)|^2 \right]\psi_+(z)$$

$$- \frac{\sqrt{N}\lambda}{2}(\alpha+\alpha^*)\cos(2z)\psi_-(z) - \sqrt{N}\lambda_\mathrm{ex}(\alpha+\alpha^*)\cos(2z)\psi_+(z)$$

where we scaled and shifted position variable $z \to z/\omega_\mathrm{L} + \pi/2\omega_\mathrm{L}$

Cumulant expansion approach

In deep optical lattice, when $V \gg \omega_{\mathrm{R}}$, condensates are well described by sum of Gaussians residing in individual lattice wells

$$\psi_\nu(t,z) = \frac{\gamma_\nu(t)}{\pi^{1/4}\sqrt{\sigma_\nu(t)}} \exp\left(-\frac{z^2}{2\sigma_\nu(t)^2} + i\eta_\nu(t)z^2\right)$$

Normalization condition: $|\gamma_-(t)|^2 + |\gamma_+(t)|^2 = 1$

Simplifying assumptions: $g_{\nu\nu'} = g$, $\sigma_\pm = \sigma$, $\eta_\pm = \eta$

Cumulant expansion leads to $\dot{\sigma} = 4\omega_{\mathrm{R}}\eta\sigma$ and EOMs

$$\dot{\alpha} = -i\partial_{\alpha^*}E - \Gamma_{\mathrm{m}}\alpha, \quad \dot{\gamma}_\nu = -i\partial_{\gamma_\nu^*}E, \quad (4\omega_{\mathrm{R}})^{-1}\ddot{\sigma} = -\partial_\sigma E$$

Cumulant expansion approach

Effective potential:

$$E[\alpha,\gamma_-,\gamma_+,\sigma] = \Omega_{\mathrm{m}}|\alpha|^2 + \frac{\Omega_{\mathrm{a}}}{2}\left(|\gamma_+|^2 - |\gamma_-|^2\right) + \frac{\omega_{\mathrm{R}}}{2\sigma^2} - \frac{V}{2}e^{-\sigma^2} + \frac{Ng}{\sqrt{8\pi}\sigma} - \sqrt{N}\lambda(\alpha+\alpha^*)\left(\chi|\gamma_+|^2 + \mathrm{Re}\{\gamma_+^*\gamma_-\}\right)e^{-\sigma^2}$$

where $\lambda_{\mathrm{ex}} = \chi\lambda$

Membrane amplitude, with $\gamma_+ = \gamma$ and $\gamma_- = \sqrt{1-\gamma^2}$:

$$\alpha_0(\gamma,\sigma) = \frac{\sqrt{N}\lambda}{\Omega_{\mathrm{m}} - i\Gamma_{\mathrm{m}}}\left[\chi\gamma^2 + \gamma\sqrt{1-\gamma^2}\right]e^{-\sigma^2}$$

Cumulant expansion approach

Effective potential becomes:

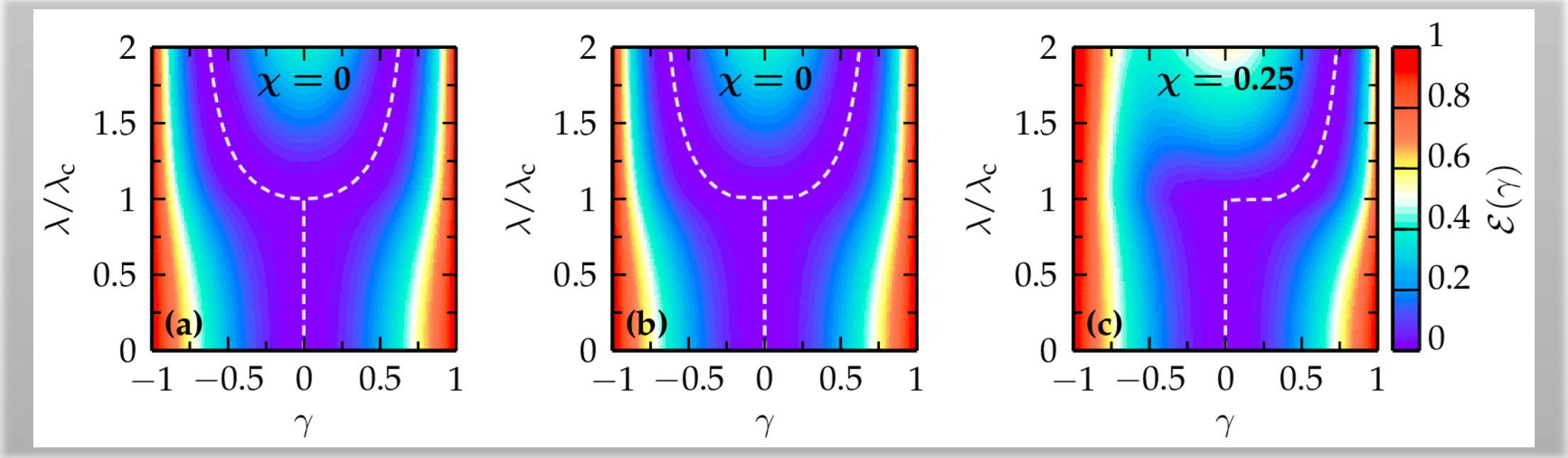
$$E[\gamma,\sigma] = \frac{\Omega_{\mathrm{a}}}{2}\left(2\gamma^2-1\right) + \frac{\omega_{\mathrm{R}}}{2\sigma^2} - \frac{V}{2}e^{-\sigma^2} + \frac{Ng}{\sqrt{8\pi}\sigma} - \frac{N\lambda^2}{\Omega'_{\mathrm{m}}}\left[\chi\gamma^2+\gamma\sqrt{1-\gamma^2}\right]^2 e^{-2\sigma^2}$$

where $\Omega'_{\mathrm{m}} = \Omega_{\mathrm{m}} + \Gamma^2_{\mathrm{m}}/\Omega_{\mathrm{m}}$, with normalized potential energy:

$$\mathcal{E}(\gamma) = \frac{E(\gamma)-E(\gamma_0)}{\max_{\bar{\gamma}}\left[E(\bar{\gamma})-E(\gamma_0)\right]}, \quad E(\gamma) = E[\gamma,\sigma_0(\gamma)]$$

exhibiting minimum $\mathcal{E}(\gamma_0) = 0$

NQPTs of 1st and 2nd order



In panels (a) and (b), a symmetric coupling ($\chi = 0$) is considered with $\Omega_a = 50\omega_R$ and $\Omega_a = 5000\omega_R$, respectively. In (a), the system exhibits a second-order phase transition, whereas the system undergoes a first-order phase transition in (b). In panel (c) is considered the asymmetric coupling regime with $\chi = 0.25$ and $\Omega_a = 50\omega_R$. Here, the system always exhibits an *asymmetric* first-order phase transition, characterized by a single favored atomic polarization.

NQPTs of 1st and 2nd order

Steady-state values are determined from:

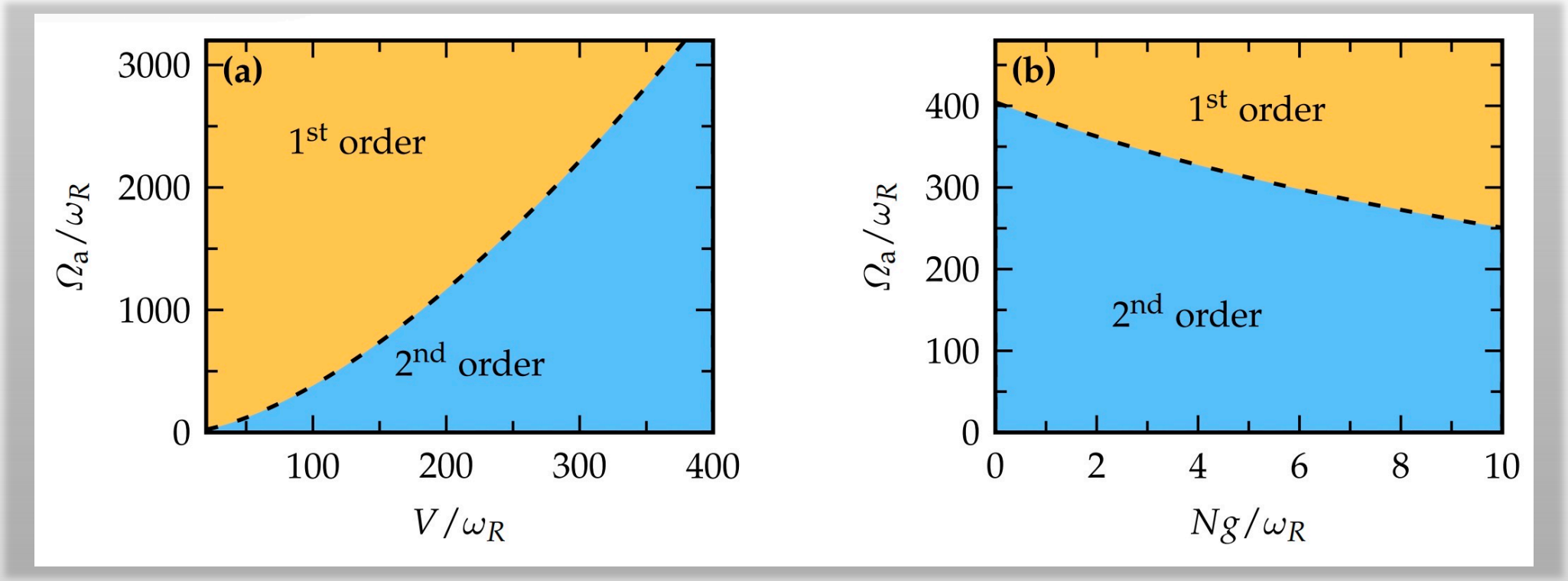
$$V e^{-\sigma_0^2}\sigma_0 + \frac{4N\lambda^2}{\Omega'_{\mathrm{m}}}\left[\chi\gamma_0^2 + \gamma_0\sqrt{1-\gamma_0^2}\right]^2 e^{-2\sigma_0^2}\sigma_0 = \frac{\omega_{\mathrm{R}}}{\sigma_0^3} + \frac{Ng}{\sqrt{8\pi}\sigma_0^2}$$

$$\gamma_0\left[N\lambda^2\left(\chi\gamma_0 + \sqrt{1-\gamma_0^2}\right)\left(2\chi\gamma_0\sqrt{1-\gamma_0^2} + 1 - 2\gamma_0^2\right) - \Omega_{\mathrm{a}}\Omega'_{\mathrm{m}}e^{2\sigma_0^2}\right] = 0$$

Equation for critical coupling λ_{s2} for second-order NQPT:

$$\omega_{\mathrm{R}} + \frac{Ng}{\sqrt{8\pi}}\sqrt{\log\frac{\sqrt{N}\lambda_{\mathrm{s2}}}{\Omega_{\mathrm{a}}\Omega'_{\mathrm{m}}}} = V\log\frac{\Omega_{\mathrm{a}}\Omega'_{\mathrm{m}}}{\sqrt{N}\lambda_{\mathrm{s2}}}\left(\log\frac{\sqrt{N}\lambda_{\mathrm{s2}}}{\Omega_{\mathrm{a}}\Omega'_{\mathrm{m}}}\right)^2$$

Symmetric coupling regime $\chi = 0$



The phase diagram of the NQPT is shown as a function of the atomic transition frequency Ω_a and (a) the lattice depth V or (b) the interaction strength Ng. While the phase transition is a continuous PT in the blue regions, it is a discontinuous PT in the orange regions. The dashed curves show the critical transition frequency $\Omega_a = \Omega_c$. The fixed parameter in (a) is the interaction strength $Ng = \omega_R$ and in (b) the potential depth $V = 100\omega_R$.

Back to mean-field equations of motion

- adiabatic elimination of excited state and light field
- mean-field equations of motion

$$i\partial_t\alpha = \big(\Omega_{\mathrm{m}} - i\Gamma_{\mathrm{m}}\big)\alpha - \int dz\,\sin(2z)\left[\frac{\sqrt{N}\lambda}{2}\big(\psi_+^*(z)\psi_-(z) + \psi_-^*(z)\psi_+(z)\big) + \sqrt{N}\lambda_{\mathrm{ex}}|\psi_+(z)|^2\right],$$

$$i\partial_t\psi_-(z) = \left[-\frac{\Omega_{\mathrm{a}}}{2} - \omega_{\mathrm{R}}\partial_z^2 + V_-(z) + N\sum_{\nu=\pm} g_{-\nu}|\psi_\nu(z)|^2\right]\psi_-(z) - \frac{\sqrt{N}\lambda}{2}\sin(2z)\big(\alpha+\alpha^*\big)\psi_+(z),$$

$$i\partial_t\psi_+(z) = \left[\frac{\Omega_{\mathrm{a}}}{2} - \omega_{\mathrm{R}}\partial_z^2 + V_+(z) + N\sum_{\nu=\pm} g_{+\nu}|\psi_\nu(z)|^2\right]\psi_+(z) - \frac{\sqrt{N}\lambda}{2}\sin(2z)\big(\alpha+\alpha^*\big)\psi_-(z)$$
$$- \sqrt{N}\lambda_{\mathrm{ex}}\sin(2z)\big(\alpha+\alpha^*\big)\psi_+(z),$$

- $\psi_\pm(z) = \langle\hat{\psi}_\pm(z)\rangle/\sqrt{N}, \quad \alpha = \langle\hat{a}\rangle/\sqrt{N}$

Back to mean-field equations of motion

- membrane almost instantaneously relaxes to its steady state

$$\bar{\alpha} = \frac{1}{\Omega_{\mathrm{m}} - i\Gamma_{\mathrm{m}}} \int dz \sin(2z) \left[\sqrt{N}\lambda_{\mathrm{ex}} |\psi_+(z)|^2 + \frac{\sqrt{N}\lambda}{2} \big(\psi_+^*(z)\psi_-(z) + \psi_-^*(z)\psi_+(z) \big) \right],$$

- coupled equations for the two condensates

$$i\partial_t \psi_-(z) = \left[-\frac{\Omega_{\mathrm{a}}}{2} - \omega_{\mathrm{R}}\partial_z^2 + V_-(z) + N \sum_{\nu=\pm} g_{-\nu} |\psi_\nu(z)|^2 \right] \psi_-(z) - \frac{\sqrt{N}\lambda}{2} \sin(2z) \big(\bar{\alpha} + \bar{\alpha}^* \big) \psi_+(z),$$

$$\begin{aligned} i\partial_t \psi_+(z) = {} & \left[\frac{\Omega_{\mathrm{a}}}{2} - \omega_{\mathrm{R}}\partial_z^2 + V_+(z) + N \sum_{\nu=\pm} g_{+\nu} |\psi_\nu(z)|^2 \right] \psi_+(z) - \frac{\sqrt{N}\lambda}{2} \sin(2z) \big(\bar{\alpha} + \bar{\alpha}^* \big) \psi_-(z) \\ & - \sqrt{N}\lambda_{\mathrm{ex}} \sin(2z) \big(\bar{\alpha} + \bar{\alpha}^* \big) \psi_+(z), \end{aligned}$$

Back to mean-field equations of motion

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- $\psi_\pm(z,t) = e^{-i\mu t} \psi_\pm(z), \quad \int dz \left(|\psi_-(z)|^2 + |\psi_+(z)|^2 \right) = 1, \quad V_\pm(z) = 0$

Back to mean-field equations of motion

- membrane almost instantaneously relaxes to its steady state

$$\bar{\alpha} = \frac{1}{\Omega_{\mathrm{m}} - i\Gamma_{\mathrm{m}}} \int dz \sin(2z) \left[\sqrt{N}\lambda_{\mathrm{ex}} |\psi_+(z)|^2 + \frac{\sqrt{N}\lambda}{2} \left(\psi_+^*(z)\psi_-(z) + \psi_-^*(z)\psi_+(z) \right) \right],$$

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$$i\partial_t \psi_-(z) = \left[-\frac{\Omega_{\mathrm{a}}}{2} - \omega_{\mathrm{R}}\partial_z^2 + V_-(z) + N \sum_{\nu=\pm} g_{-\nu} |\psi_\nu(z)|^2 \right] \psi_-(z) - \frac{\sqrt{N}\lambda}{2} \sin(2z) \left(\bar{\alpha} + \bar{\alpha}^* \right) \psi_+(z),$$

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$$- \sqrt{N}\lambda_{\mathrm{ex}} \sin(2z) \left(\bar{\alpha} + \bar{\alpha}^* \right) \psi_+(z),$$

- $\psi_\pm(z,t) = e^{-i\mu t}\psi_\pm(z), \quad \int dz \left(|\psi_-(z)|^2 + |\psi_+(z)|^2 \right) = 1,$ $\boxed{V_\pm(z) = 0}$

Homogeneous ground steady states

- $\psi_\pm(z) = \sqrt{n_\pm/\pi}, \quad n_- + n_+ = 1, \quad \bar{\alpha} = 0$

- three possible solutions:

$$n_- = 1, \quad n_+ = 0, \quad \mu = -\frac{\Omega_\mathrm{a}}{2} + \frac{N}{\pi} g_{--}, \quad \pi\Omega_\mathrm{a}/N > g_{--} - g_{+-}$$

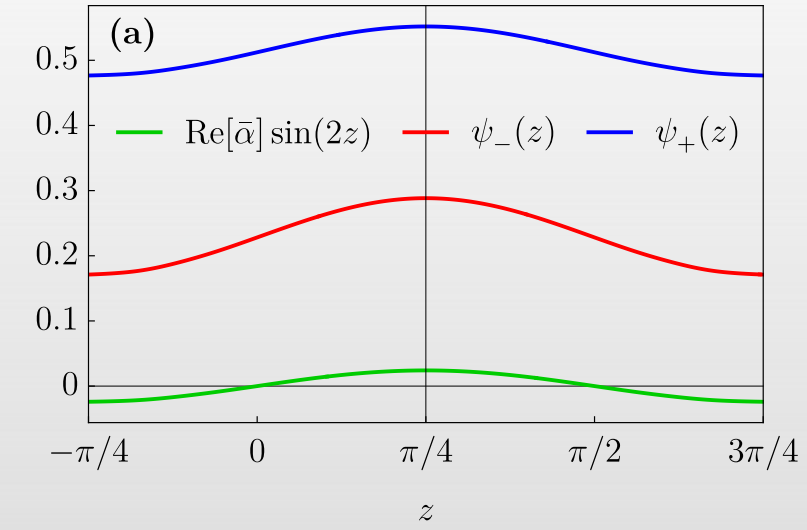
$$n_- = 0, \quad n_+ = 1, \quad \mu = \frac{\Omega_\mathrm{a}}{2} + \frac{N}{\pi} g_{++}, \quad \pi\Omega_\mathrm{a}/N < g_{+-} - g_{++}$$

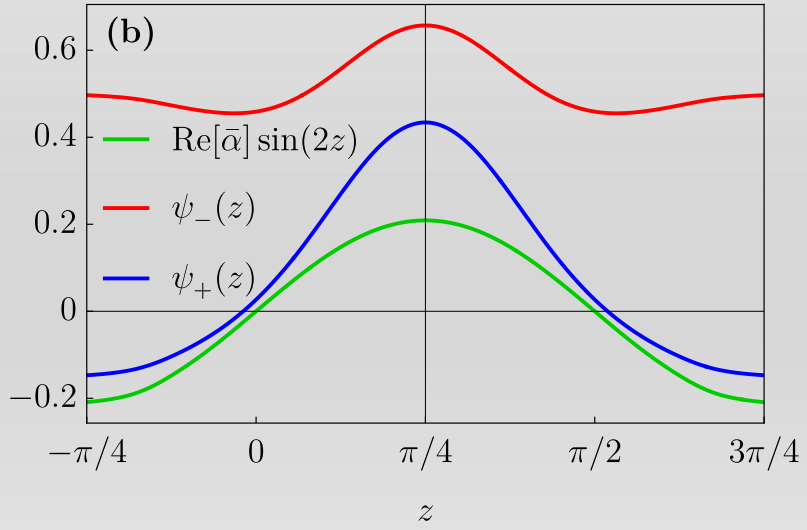
$$n_- = \frac{g_{++} - g_{+-} + \pi\Omega_\mathrm{a}/N}{g_{--} + g_{++} - 2g_{+-}}, \quad n_+ = \frac{g_{--} - g_{+-} - \pi\Omega_\mathrm{a}/N}{g_{--} + g_{++} - 2g_{+-}},$$

$$\mu = \frac{\Omega_\mathrm{a}}{2} \frac{g_{--} - g_{++}}{g_{--} + g_{++} - 2g_{+-}} + \frac{N}{\pi} \frac{g_{--}g_{++} - g_{+-}^2}{g_{--} + g_{++} - 2g_{+-}}.$$

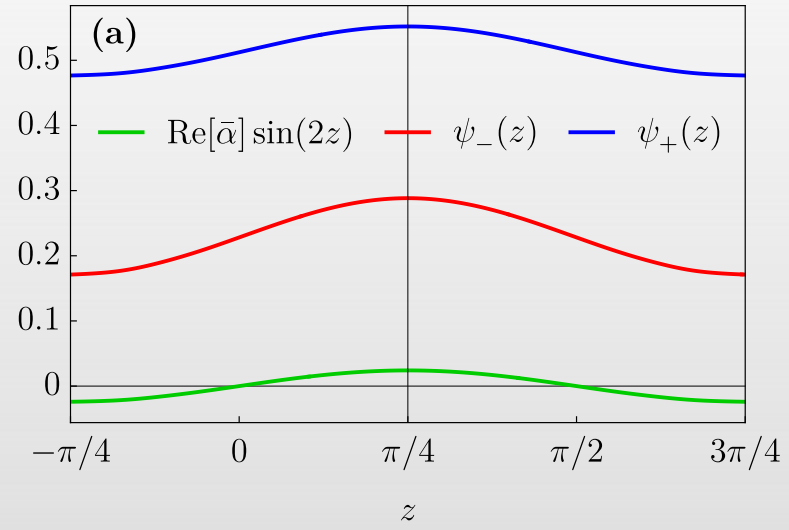
- $\delta\Omega_\mathrm{a} \equiv \Omega_\mathrm{a} - \Omega_{\mathrm{a},0}, \quad \Omega_{\mathrm{a},0} = N(g_{--} - g_{++})/(2\pi), \quad d_{-+} = (N_- - N_+)/N$

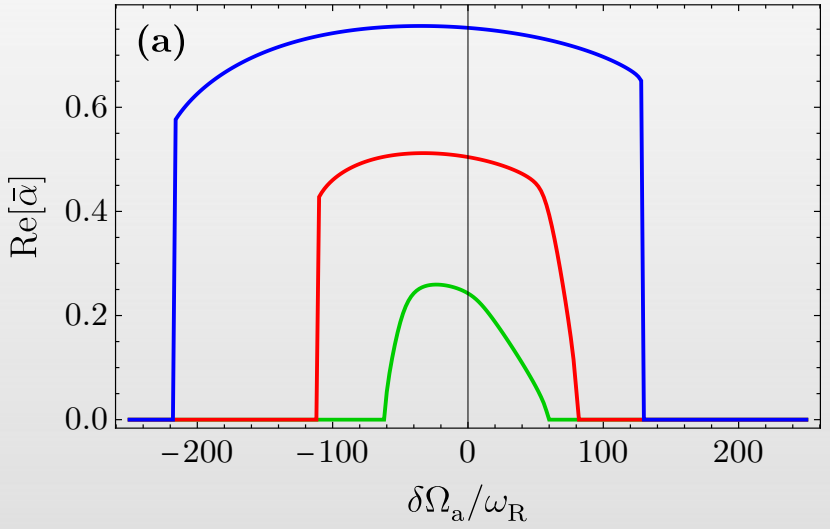
Inhomogeneous ground steady states

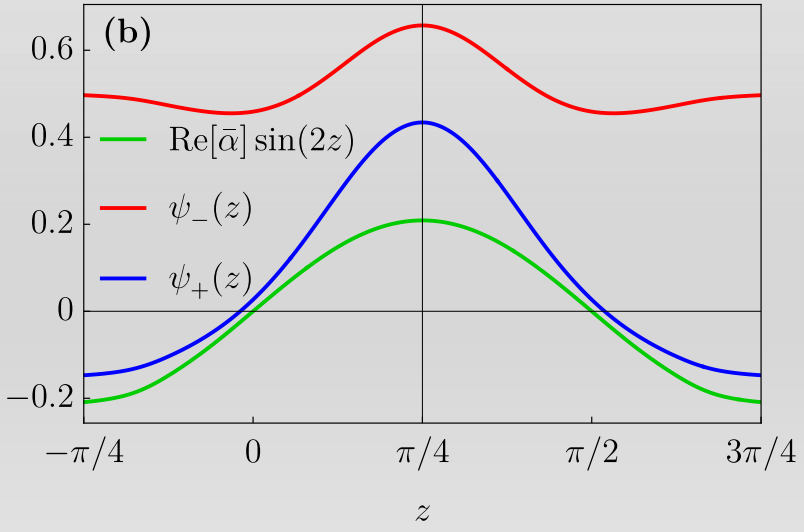


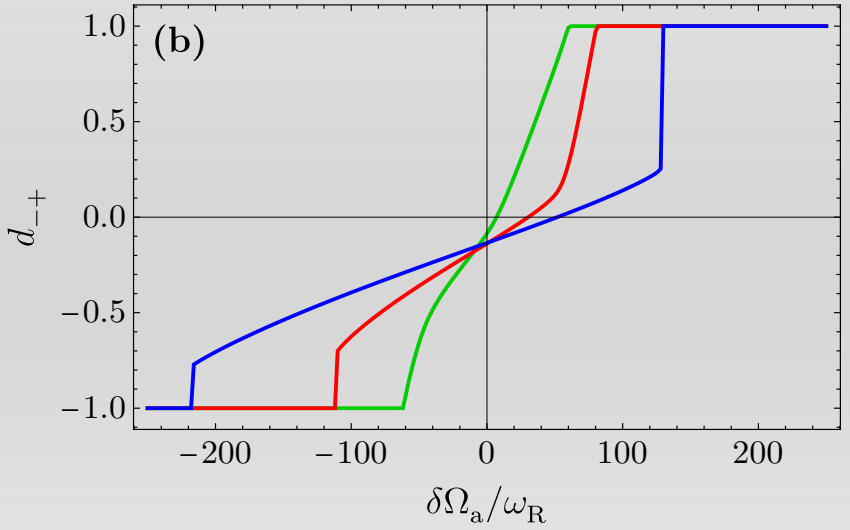


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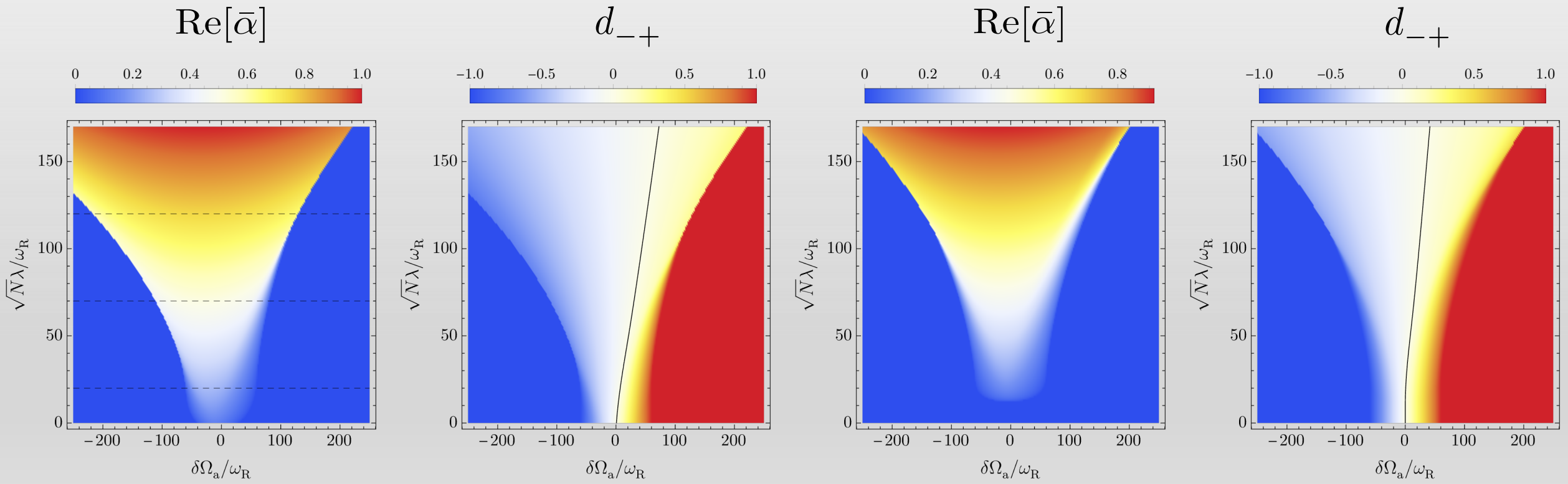






Nonequilibrium quantum phase transitions

- uniform phase for weak effective membrane-atom coupling
- self-organized density wave-like phase for stronger coupling



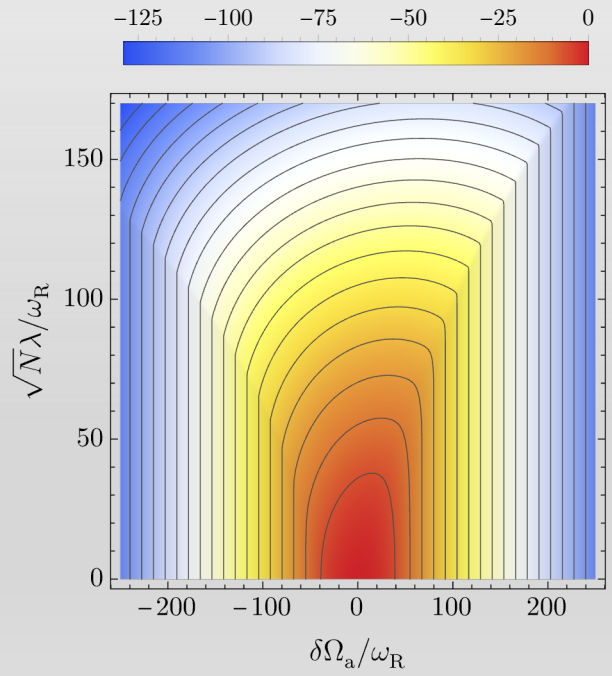
$\sqrt{N}\lambda_{\mathrm{ex}} = 40\omega_{\mathrm{R}}$

$\sqrt{N}\lambda_{\mathrm{ex}} = 25\omega_{\mathrm{R}}$

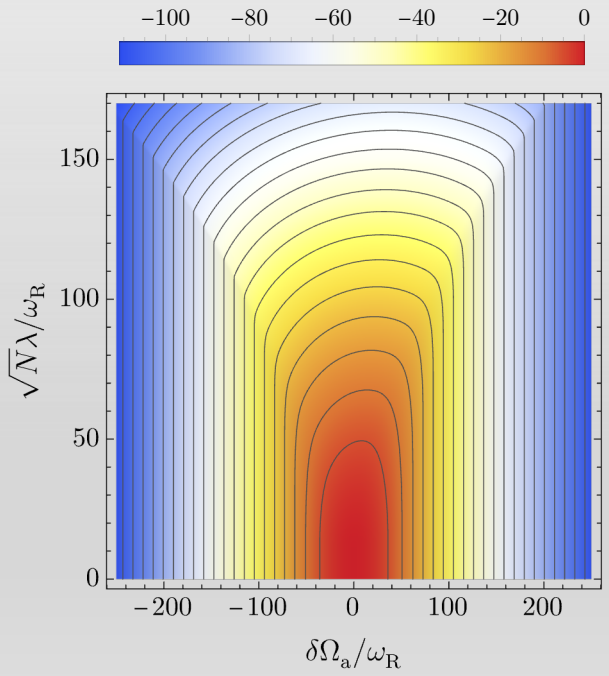
Nonequilibrium quantum phase transitions

- energy features gradient jumps for discontinuous phase transitions
- asymmetric role of couplings λ and λ_{ex}

Energy density

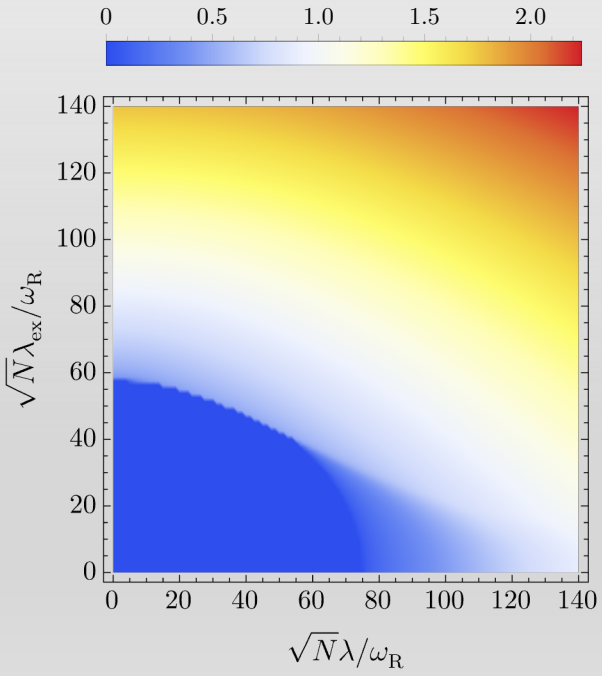


$\sqrt{N}\lambda_{\mathrm{ex}} = 40\omega_{\mathrm{R}}$

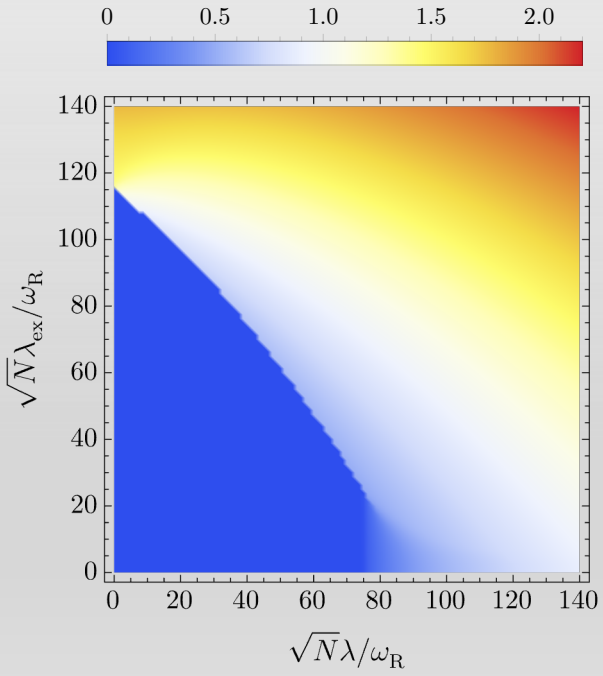


$\sqrt{N}\lambda_{\mathrm{ex}} = 25\omega_{\mathrm{R}}$

$\mathrm{Re}[\bar{\alpha}]$



$\delta\Omega_{\mathrm{a}} = -120\omega_{\mathrm{R}}$



$\delta\Omega_{\mathrm{a}} = 120\omega_{\mathrm{R}}$

Conclusions and Outlook

- this hybrid system is versatile, offering multiple tuning knobs for changing order of phase transitions
- phases can be experimentally distinguished by measuring membrane displacement
- population imbalance further characterizes homogeneous state
- contact interaction strengths $g_{\nu\nu'}$ also have an influence
- dynamics beyond nonequilibrium steady state?
- effect of quantum fluctuations?
- do phase transitions change their order?
- are there self-bound quantum droplets?

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Thanks for your attention!

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