# Mathematical methods for Quantum Phase Transitions in Optical Lattices

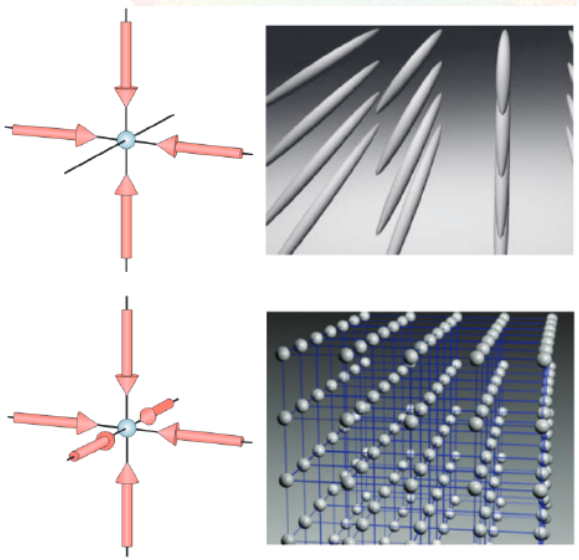
F. E. A. dos Santos

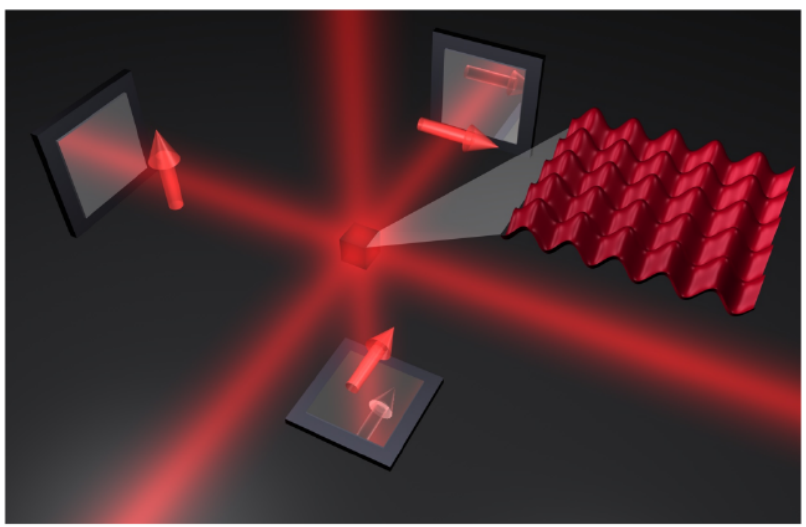
Intensive Week on Phase Transitions in Atomic and Photonic Matter



# Optical Lattices

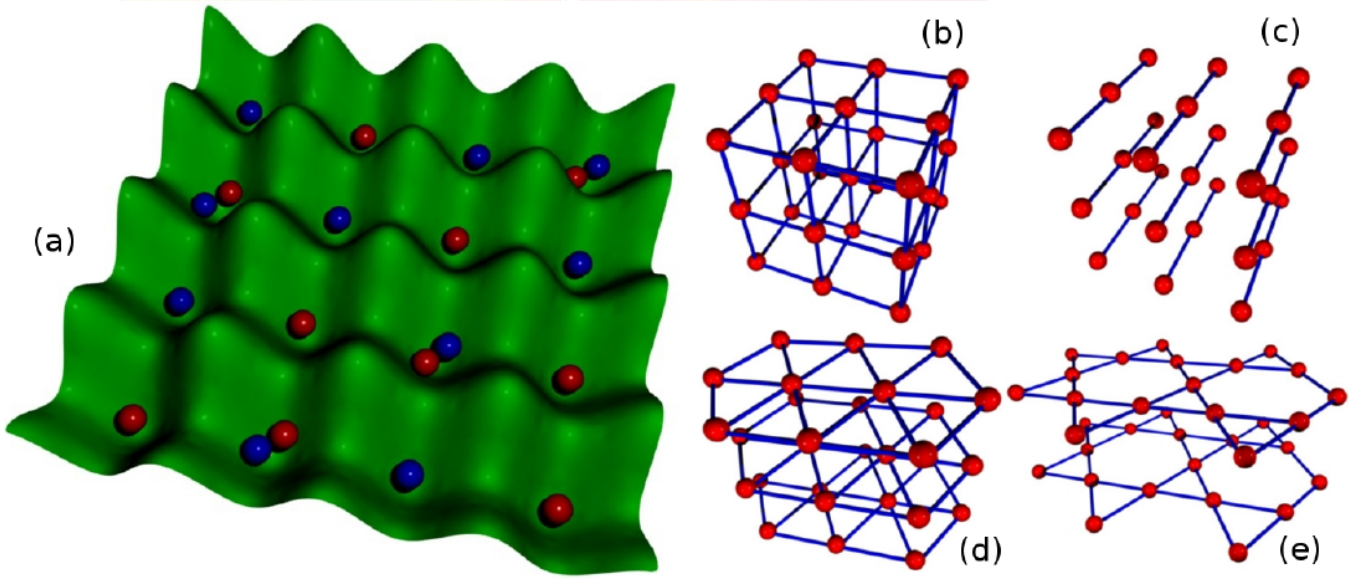
- Formed by intersecting two or more laser beams of a specific wavelength and polarization.
- Atoms or molecules placed in this pattern experience a periodic potential energy due to the varying electric field strength of the light.
- Depth and interactions can be tunned
- Can be created in one, two, or three dimensions.





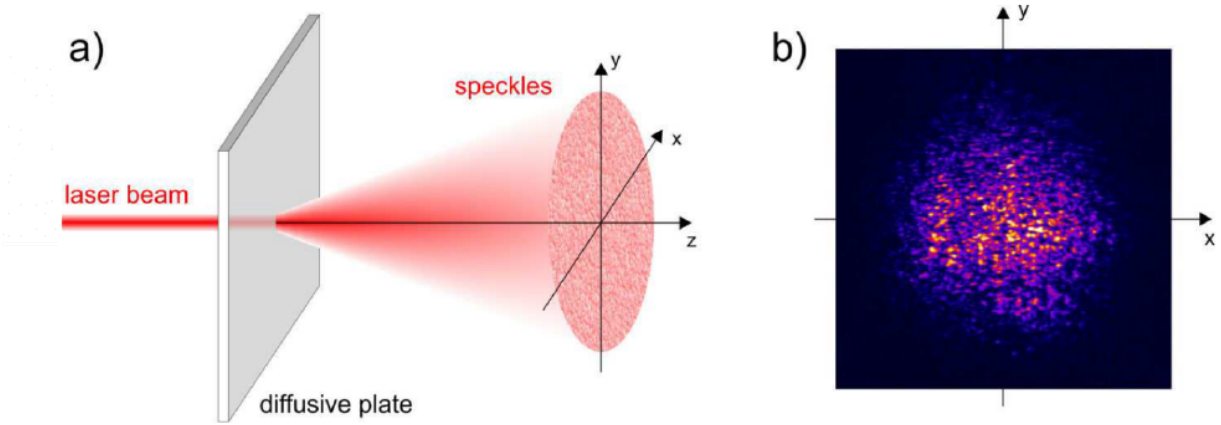
# Optical Lattices

- No disorder in principle
- Can be tailored to create artificial band structures for atoms and molecules.
- Can be used to simulate complex quantum systems, such as superconductors and topological materials.
- Floquet engineering has been used to create artificial gauge fields, topological insulators, and simulate non-equilibrium quantum systems.



# Adding Disorder

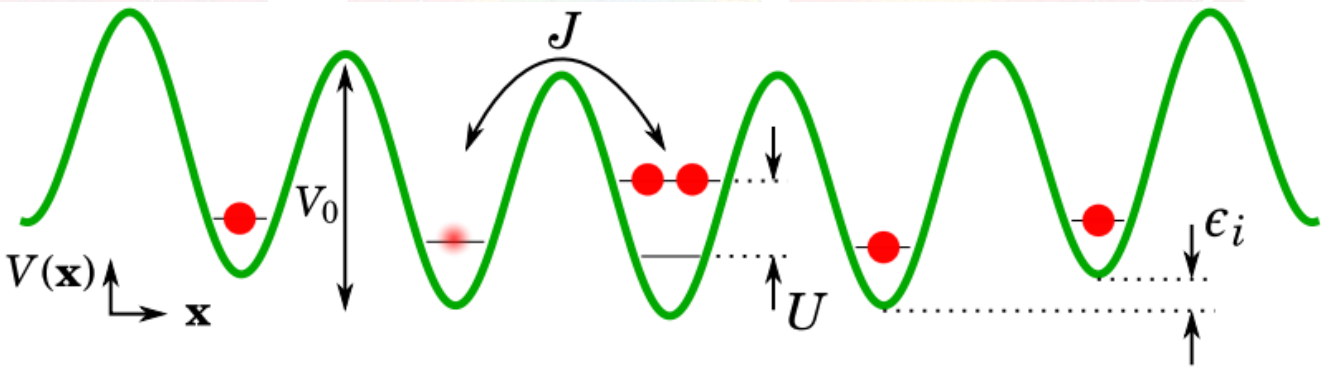
- Quasi-Periodic lattices by using incommensurate laser beams.
- This results in a lattice with long-range order but short-range disorder.
- Controlled defects introduced into the lattice by adding impurities to the atomic sample.
- External noise sources, such as acoustic or magnetic noise.
- Leads to suppression of quantum transport
- Formation of new quantum phases



# Bose-Hubbard Hamiltonian

- Bose-Hubbard Hamiltonian

$$\hat{H}_{\mathrm{BH}} = -\sum_{\langle ij \rangle} J_{ij} \hat{a}_i^\dagger \hat{a}_j + \frac{U}{2} \sum_i \hat{a}_i^\dagger \hat{a}_i^\dagger \hat{a}_i \hat{a}_i + \sum_i (\varepsilon_i - \mu) \hat{a}_i^\dagger \hat{a}_i$$




- Order parameter:

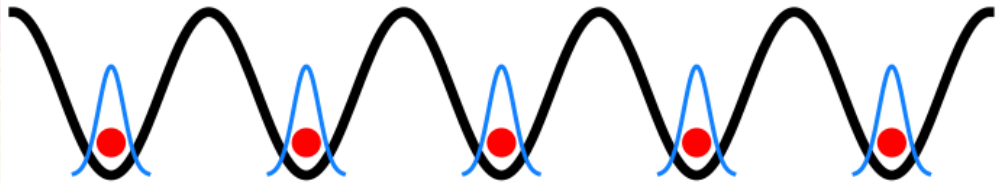
$$\Psi = \sum_i \psi_i = \sum_i \langle \hat{a}_i \rangle$$

# Bose-Hubbard Hamiltonian

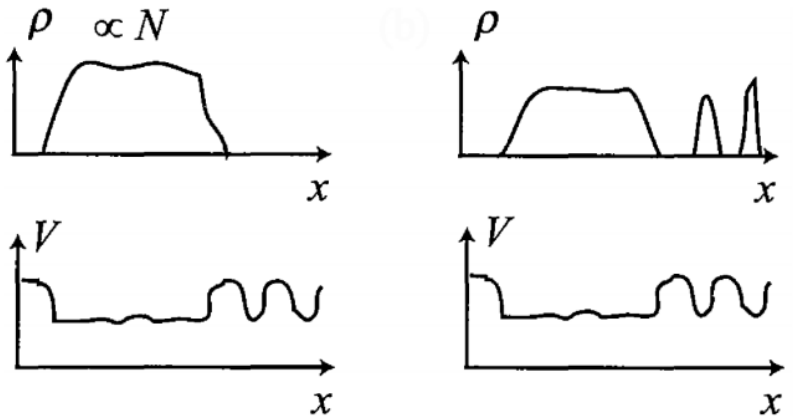
- Superfluid phase ($U/J \ll 1$, $\psi \neq 0$): Bosons are highly mobile and form a coherent state with long-range order.

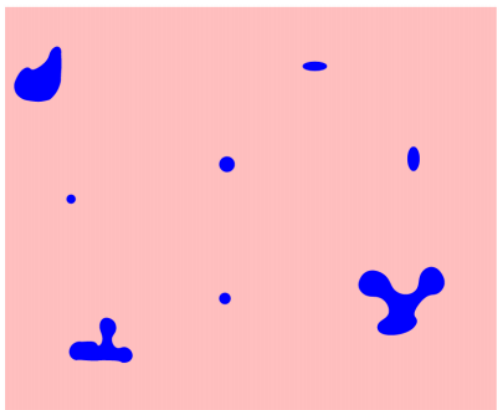


- Mott insulator phase ($U/J \gg 1$ , $\psi = 0$): Bosons are localized due to strong interactions, leading to non-conducting behavior.



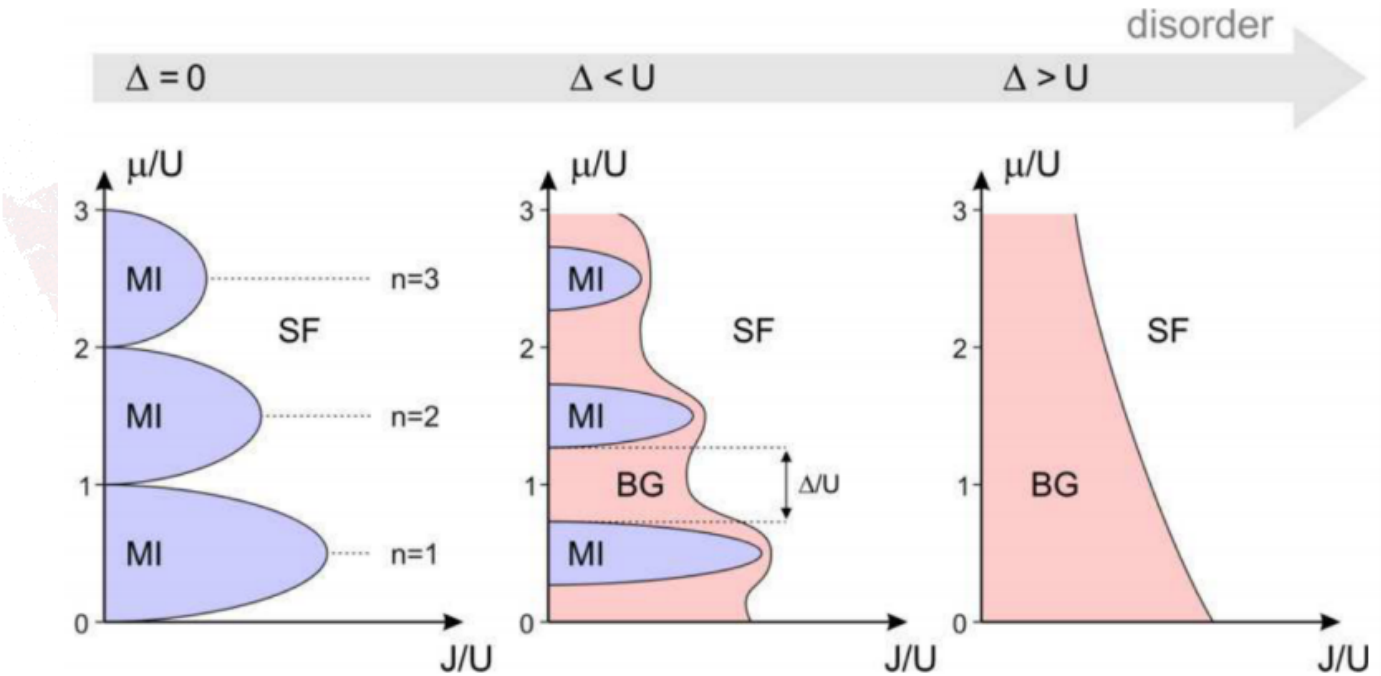
- Bose glass phase ($U/J \sim 1$): Transition region between the superfluid and Mott insulator phases.



# Phase diagram

- Week interaction expansions can only see the superfluid phase




Uniform disorder distribution: $-\Delta < \varepsilon < \Delta$

- We need a systematic strong coupling expansion for quantum lattices

# Phase Transitions

- Phase transitions involve changes in physical properties due to external parameters.
- Examples: ferromagnetic phase transition (rotational symmetry), crystal formation (translational symmetry).
- Second-order phase transitions often involve symmetry breaking.
- Leads to sudden change in physical properties (e.g., magnetic properties, state of matter)

# Landau Theory

- Landau theory of phase transitions assumes that the free energy of a system can be expanded in a power series of an order parameter:

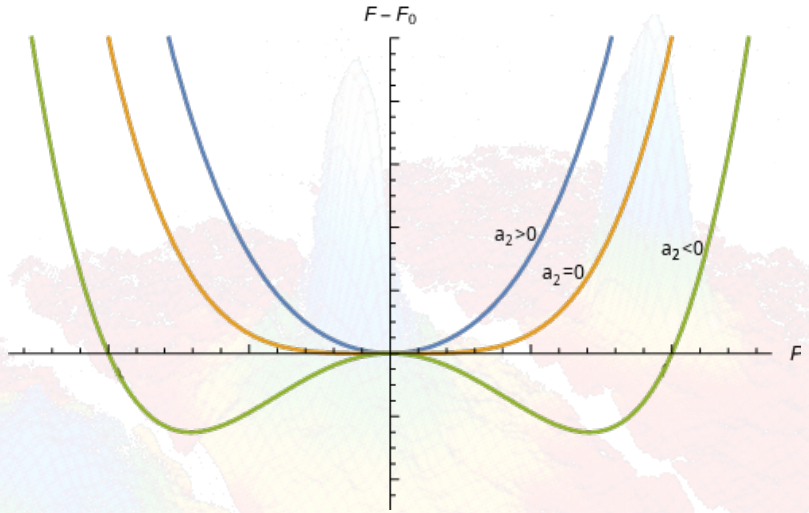
$$\Gamma = F_0 + \frac{1}{2!} a_2 P^2 + \frac{1}{4!} a_4 P^4 + \cdots$$

$$\frac{\partial \Gamma}{\partial P} = 0$$

- $F_0$ is the free energy of the symmetric phase
- $P$ is the order parameter
- $a_4$ is usually considered positive
- If $a_2$ is positive, the free energy has a minimum at $P = 0$, which corresponds to the symmetric phase.
- If $a_2$ is negative, the free energy has a minimum at $P \neq 0$, which corresponds to the broken symmetry phase.

# Landau Theory

- Landau expansion: $\Gamma = F_0 + \frac{1}{2!} a_2 P^2 + \frac{1}{4!} a_4 P^4 + \cdots$




- Universality: Behavior near phase transition is independent of microscopic details
- Critical exponents: Predicts exponents describing behavior of physical quantities near critical point
- Symmetry breaking: Relates phase transitions to symmetry breaking
- Mean-field approximation: Neglects fluctuations
- Cannot describe first-order phase transitions

# Landau Theory

- Classical case

$$\mathscr{Z} = \int dX \; e^{-\beta \mathscr{H}(X)}, \qquad F_0 = -\frac{1}{\beta} \ln(\mathscr{Z})$$

- Due to the symmetry, direct calculation for a finite system always leads to:

$$P = \langle \mathscr{P}(X) \rangle = \frac{1}{\mathscr{Z}} \int dX \; \mathscr{P}(X) \; e^{-\beta \mathscr{H}(X)} = 0$$

- Symmetry disappears due to ergodicity breakdown
- Therefore, symmetry must be explicitly broken in calculations

$$\mathscr{H}(X) \to \mathscr{H}(X) + J \; \mathscr{P}(X)$$

$$\mathscr{Z}(J) = \int dX \; e^{-\beta \mathscr{H}(X)} e^{-\beta J \mathscr{P}(X)}$$

$$= \mathscr{Z} \left[ 1 + \frac{1}{2!} (\beta J)^2 \left\langle \mathscr{P}^2 \right\rangle + \frac{1}{4!} (\beta J)^4 \left\langle \mathscr{P}^4 \right\rangle + \cdots \right]$$

- $J$ is a Lagrange multiplier used to fix $P$
- Power expansion valid in the symmetric phase

# Landau Theory

- Order parameter

$$F(J) = -\frac{1}{\beta} \ln \left[ \mathscr{Z}(J) \right]$$

$$P(J) = \langle \mathscr{P}(X) \rangle = \frac{\partial F(J)}{\partial J}$$

- In principle $P(J)$ can be inverted so that $J \equiv J(P)$
- Landau Free Energy (Effective Potential):

$$\Gamma(P) = F\left(J(P)\right) - J(P)\ P$$

$$J(P) = \frac{\partial \Gamma(P)}{\partial P}$$

- For the physical case $J = \frac{\partial \Gamma(P)}{\partial P} = 0$, i.e., $\Gamma$ must be at its minimum.

# Effective Potential

- $F$ is the generator of the cummulants

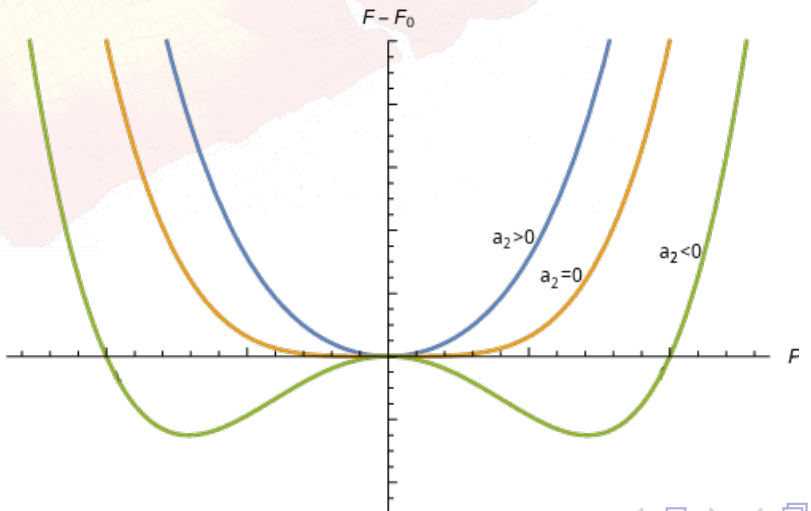
$$
\begin{aligned}
F(J) &= -\frac{1}{\beta}\ln\left[\mathscr{Z}(J)\right] \\
&= F_0 - \frac{\beta}{2!}\left\langle\mathscr{P}^2\right\rangle J^2 + \frac{\beta}{4!}\left(\left\langle\mathscr{P}^4\right\rangle - 3\left\langle\mathscr{P}^2\right\rangle^2\right)J^4 + \cdots \\
P(J) &= -\beta\left\langle\mathscr{P}^2\right\rangle J + \frac{\beta}{3!}\left(\left\langle\mathscr{P}^4\right\rangle - 3\left\langle\mathscr{P}^2\right\rangle^2\right)J^3 \\
J(P) &= -\frac{1}{\beta\left\langle\mathscr{P}^2\right\rangle}P - \frac{\left\langle\mathscr{P}^4\right\rangle - 3\left\langle\mathscr{P}^2\right\rangle^2}{3!\beta^2\left\langle\mathscr{P}^2\right\rangle^4}P^4 + \cdots \\
\Gamma(P) &= F_0 + \frac{1}{2!}\frac{1}{\beta\left\langle\mathscr{P}^2\right\rangle}P^2 + \frac{1}{4!}\frac{\left\langle\mathscr{P}^4\right\rangle - 3\left\langle\mathscr{P}^2\right\rangle^2}{\beta\left\langle\mathscr{P}^2\right\rangle^4}P^4 + \cdots
\end{aligned}
$$

# Landau Theory

- Therefore $\Gamma = F_0 + \frac{1}{2!} a_2 P^2 + \frac{1}{4!} a_4 P^4 + \cdots$, where

$$a_2 = \frac{1}{\beta \left\langle \mathscr{P}^2 \right\rangle}; \qquad a_4 = \frac{\left\langle \mathscr{P}^4 \right\rangle - 3 \left\langle \mathscr{P}^2 \right\rangle^2}{\beta \left\langle \mathscr{P}^2 \right\rangle^4}$$

- Phase transition happens when $a_2 = 0$, i.e., $\left\langle \mathscr{P}^2 \right\rangle \to \infty$
- Analytic continuation for both $a_2$ and $a_4$ are necessary to get results in the symmetry-broken phase
- Landau proposed that $a_2$ and $a_4$ are analytical functions of the system parameters



# Long-range correlations

- Modularity:

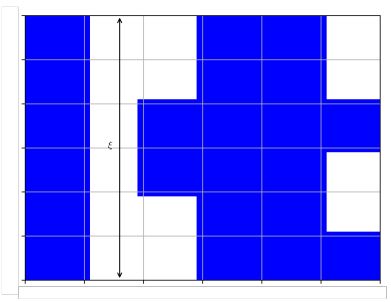
$$\mathscr{P} = \sum_I \mathscr{P}(\mathbf{r}_I)$$

$$\frac{\langle \mathscr{P}^2 \rangle}{N} = \sum_{I,m} \langle \mathscr{P}(\mathbf{r}_I)\mathscr{P}(\mathbf{r}_m) \rangle$$

- Correlations function in translational invariance: $C(\mathbf{r}_I, \mathbf{r}_m) = C(\mathbf{r}_I - \mathbf{r}_m)$

$$\frac{\langle \mathscr{P}^2 \rangle}{N} = \frac{1}{N} \sum_{I,m} C(\mathbf{r}_I - \mathbf{r}_m) = \sum_I C(\mathbf{r}_I)$$

- Long-range correlation $\xi \to \infty$ leads to $\langle \mathscr{P}^2 \rangle \to \infty$ and $a_2 \to 0$
- Critical exponent $\xi \sim |g - g_c|^{-\nu}$



# Quantum Phase Transition

- Quantum partition function

$$\mathscr{Z} = \mathrm{Tr}\left[e^{-\beta\hat{H}}\right] = \oint_{\phi(0)=\phi(\hbar\beta)} \mathscr{D}[\phi] \exp\left[-\frac{1}{\hbar}\int_0^{\hbar\beta} d\tau\ L(\phi, \partial\phi)\right]$$

- $L$ is the Lagrangean function in imaginary time $t = i\ \tau$.
- For the zero-temperature case $\beta \to \infty$ and a d-dimensional quantum system behaves as a (d+1)-dimensional classical system.
- All the ideas about phase transitions can be borrowed from the classical case.
- Phase transition occurs due to changes in a non-thermal parameter.
- Driven by quantum fluctuations, even at zero temperature.
- Critical exponent $z$ associated with the "time" direction $\tau \sim \Delta^{-1} \sim |g - g_c|^{z\nu}$

# Quantum Phase Transition

- Symmetry broken Hamiltonian

$$\hat{H}_{\mathrm{BH}}(\tau) = \hat{H}_{\mathrm{BH}} + \sum_i \left[ j_i^*(\tau)\hat{a}_i + j_i(\tau)\hat{a}_i^\dagger \right].$$

- The imaginary-time evolution operator ($\tau = it$)

$$-\frac{\partial \hat{U}(\tau,\tau_0)}{\partial \tau} = \hat{H}_{\mathrm{BH}}(\tau)\hat{U}(\tau,\tau_0)$$

$$\hat{U}(\tau,\tau_0) = \hat{T}\, e^{-\int_{\tau_0}^{\tau} d\tau' \hat{H}_{\mathrm{BH}}(\tau')}$$

$$\mathscr{Z}[j^*(\tau), j(\tau)] = \mathrm{Tr}\hat{U}(\beta,0)$$

- Born series

$$\begin{aligned}\hat{U}(\tau,\tau_0) &= \hat{1} - \int_{\tau_0}^{\tau} d\tau_1 \hat{H}(\tau_1) + \int_{\tau_0}^{\tau} d\tau_1 \int_{\tau_0}^{\tau_1} d\tau_2 \hat{H}(\tau_2)\hat{H}(\tau_1) \\ &\quad - \int_{\tau_0}^{\tau} d\tau_1 \int_{\tau_0}^{\tau_1} d\tau_2 \int_{\tau_0}^{\tau_2} d\tau_3 \hat{H}(\tau_3)\hat{H}(\tau_2)\hat{H}(\tau_1) + \cdots\end{aligned}$$

- Semi-group property

$$\hat{U}(\tau,\tau_0) = \hat{U}(\tau,\tau')\hat{U}(\tau',\tau_0), \qquad \text{if } \tau \geq \tau' \geq \tau_0$$

$$\hat{U}(\beta,0) = \hat{U}(\beta,\tau+\Delta)\hat{U}(\tau+\Delta,\tau-\Delta)\hat{U}(\tau-\Delta,0)$$

$$\frac{\delta}{\delta j_i^*(\tau)}\hat{U}(\beta,0) = \hat{U}(\beta,\tau+\Delta)\frac{\delta \hat{U}(\tau+\Delta,\tau-\Delta)}{\delta j_i^*(\tau)}\hat{U}(\tau-\Delta,0)$$

$$\frac{\delta \hat{U}(\tau+\Delta,\tau-\Delta)}{\delta j_i^*(\tau)} = -\hat{a}_i$$

$$+\int_{\tau-\Delta}^{\tau+\Delta} d\tau_1 \left[\Theta(\tau-\tau_1)\hat{a}_i\hat{H}(\tau_1) + \Theta(\tau_1-\tau)\hat{H}(\tau_1)\hat{a}_i\right] - \cdots .$$

- Taking the limit $\Delta \to 0$:

$$\frac{\delta \hat{U}(\beta,0)}{\delta j_i^*(\tau)} = -\hat{U}(\beta,\tau)\hat{a}_i\hat{U}(\tau,0) = -\hat{U}(\beta,0)\hat{a}_i(\tau)$$

$$\hat{a}_i(\tau) = \hat{U}^{-1}(\tau,0)\hat{a}_i\hat{U}(\tau,0)$$

- Taking more functional derivatives:

$$\frac{\delta^{(n+m)}\hat{U}(\beta,0)}{\delta j^*_{i_1}(\tau_1)\cdots\delta j^*_{i_n}(\tau_n)\delta j_{i'_1}(\tau'_1)\cdots\delta j_{i'_m}(\tau'_m)} =$$

$$= \hat{U}(\beta,0)\,\hat{T}\left[\hat{a}_{i_1}(\tau_1)\cdots\hat{a}_{i_n}(\tau_n)\hat{a}^\dagger_{i'_1}(\tau'_1)\cdots\hat{a}^\dagger_{i'_m}(\tau'_m)\right].$$

$$G^{(n,m)}\left(\tau_1,i_1;\cdots;\tau_n,i_n|\tau'_1,i'_1;\cdots;\tau'_m,i'_m\right) =$$

$$= Z[j^*,j]^{-1}\left.\frac{\delta^{(n+m)}Z[j^*,j]}{\delta j^*_{i_1}(\tau_1)\cdots\delta j^*_{i_n}(\tau_n)\delta j_{i'_1}(\tau'_1)\cdots\delta j_{i'_m}(\tau'_m)}\right|_{j^*=j=0}.$$

# Quantum Phase Transition

- The source-dependent thermodynamic potentials

$$F[j^*,j] = -\frac{1}{\beta}\ln Z[j^*,j] = -\frac{1}{\beta}W[j^*,j]$$

- Order- parameter field

$$\psi_i(\tau) = \langle \hat{a}_i(\tau)\rangle = \beta\frac{\delta F}{\delta j_i^*(\tau)}$$

$$\psi_i^*(\tau) = \left\langle \hat{a}_i^\dagger(\tau)\right\rangle = \beta\frac{\delta F}{\delta j_i(\tau)}$$

- Effective Action

$$\Gamma[\psi^*,\psi] = F - \frac{1}{\beta}\sum_i \int_0^\beta d\tau[\psi_i^*(\tau)j_i(\tau) + \psi_i(\tau)j_i^*(\tau)]$$

$$\frac{\delta\Gamma[\psi^*,\psi]}{\delta\psi_i(\tau)} = 0$$

$$\frac{\delta\Gamma[\psi^*,\psi]}{\delta\psi_i^*(\tau)} = 0$$

# Perturbation Theory

- Perturbation:

$$\hat{H}_{\mathrm{BH}}(\tau) = \hat{H}_0(\tau) + \hat{H}_{\mathrm{h}}$$

- Interaction Picture

$$\hat{U}(\tau,0) = \hat{U}_0(\tau,0)\hat{U}_{\mathrm{I}}(\tau,0)$$

$$\hat{U}_0(\tau,0) = \hat{T}\, e^{-\int_0^\tau d\tau' \hat{H}_0(\tau')}$$

$$-\frac{\partial \hat{U}_{\mathrm{I}}(\tau,\tau_0)}{\partial \tau} = \hat{H}_{\mathrm{h}}^{\mathrm{I}}(\tau)\hat{U}(\tau,\tau_0)$$

$$\hat{H}_{\mathrm{h}}^{\mathrm{I}}(\tau) = \hat{U}_0^{-1}(\tau,0)\hat{H}_{\mathrm{h}}\hat{U}_0(\tau,0)$$

$$\hat{U}_I(\tau,0) = \hat{T}\, e^{-\int_0^\tau d\tau' \hat{H}_{\mathrm{h}}^{\mathrm{I}}(\tau')}$$

- Partition function

$$Z[j^*,j] = \mathrm{Tr}\left[\hat{U}_0(\tau;0)\,\hat{T}\, e^{-\int_0^\tau d\tau' \hat{H}_{\mathrm{h}}^{\mathrm{I}}(\tau')}\right]$$

$$Z[j^*,j] = e^{\sum_{ab} J_{ab} \int_0^\beta d\tau \frac{\delta^2}{\delta j_a^*(\tau)\delta j_b(\tau)}} Z_0[j^*,j]$$

# Lattice Diagrammatics

- Unperturbed system:

$$Z_0[j^*,j] = \prod_i Z_i[j_i^*,j_i]$$

$$W_0[j^*,j] = -\beta F_0[j^*,j] = -\beta \sum_i W_i[j_i^*,j_i]$$

$$\Gamma_0[j^*,j] = \sum_i \Gamma_i[j_i^*,j_i]$$

- Expansion in $j^*$and $j$:

$$W_i[j^*,j] = W_i^{(0)} + \int_0^\beta d\tau \int_0^\beta d\tau' W_i^{(2)}(\tau;\tau') j_i^*(\tau)\delta j_i(\tau')$$
$$+\frac{1}{2!^2}\int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \int_0^\beta d\tau_1' \int_0^\beta d\tau_2' W_i^{(4)}(\tau_1,\tau_2;\tau_1',\tau_2')$$
$$\times j_i(\tau_1) j_i(\tau_2) j_i^*(\tau_1') j_i^*(\tau_2') + \cdots$$

# Lattice Diagrammatics

- Effective Action:

$$Z_0[j^*,j] = \prod_i Z_i[j_i^*,j_i]$$

$$W_0[j^*,j] = -\beta F_0[j^*,j] = \sum_i W_i[j_i^*,j_i]$$

$$\Gamma_0[j^*,j] = \sum_i \Gamma_i[j_i^*,j_i]$$

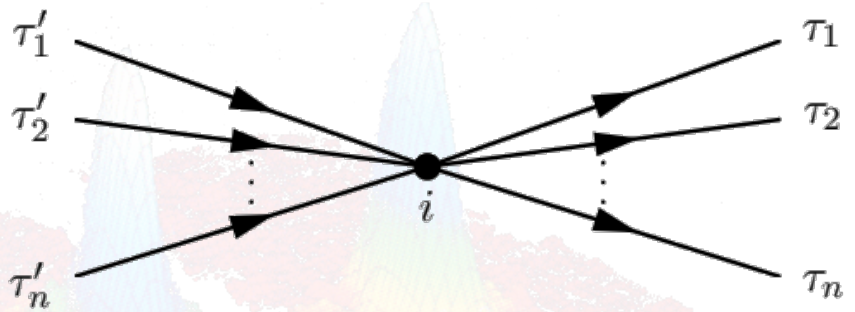
- Expansion in $j^*$and $j$:

$$W_i[j^*,j] = W_i^{(0)} + \int_0^\beta d\tau \int_0^\beta d\tau' W_i^{(2)}(\tau;\tau') j_i^*(\tau)\delta j_i(\tau')$$
$$+\frac{1}{2!^2}\int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \int_0^\beta d\tau_1' \int_0^\beta d\tau_2' W_i^{(4)}(\tau_1,\tau_2;\tau_1',\tau_2')$$
$$\times j_i(\tau_1) j_i(\tau_2) j_i^*(\tau_1') j_i^*(\tau_2') + \cdots$$

# Lattice Diagrammatics

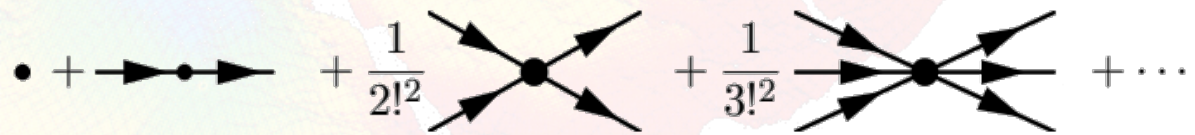
- Connected functions local correlation functions[1]:

$$W_{0i}^{(2n)}(\tau_i, \cdots, \tau_n; \tau_1', \cdots, \tau_n') =$$



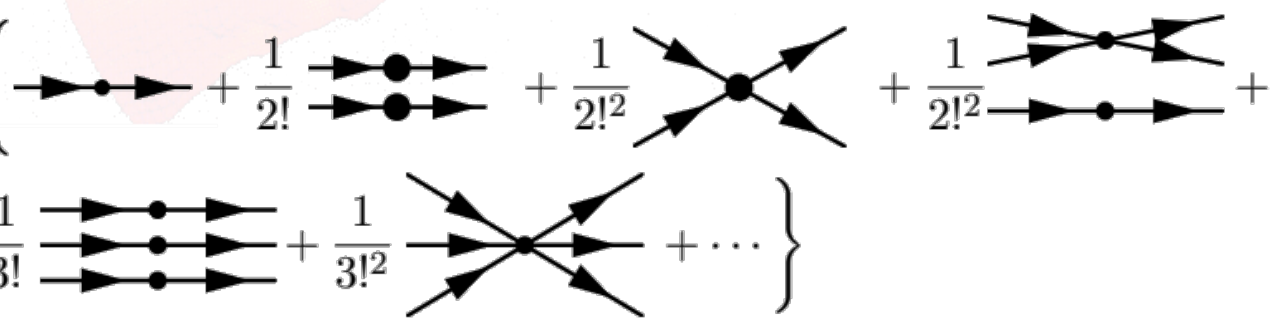

- Generator of local connected functions:

$$W_0[j^*, j] =$$




- Local Partition function

$$Z^{(0)}\{j^*, j\} = e^{\bullet}$$




[1] Walter Metzner. Phys. Rev. B 43, 8549, 1991

# Lattice Diagrammatics

$$Z[j^*,j] = e^{\sum_{ab} J_{ab} \int_0^\beta d\tau \frac{\delta^2}{\delta j_a^*(\tau)\delta j_b(\tau)}} Z_0[j^*,j]$$

$$= \left(1 + \sum_{ab} J_{ab} \int_0^\beta d\tau \frac{\delta^2}{\delta j_a^*(\tau)\delta j_b(\tau)} + \cdots\right) Z_0[j^*,j]$$

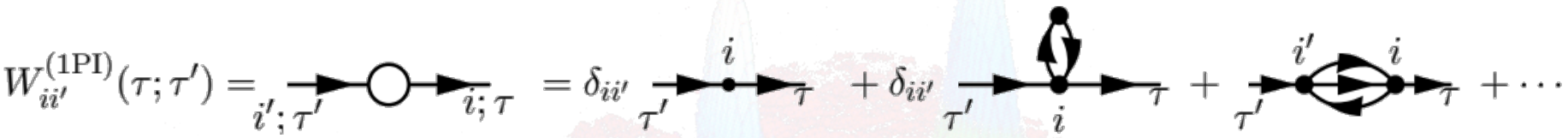
$$Z\{j^*,j\} = e^{\bullet}\Big\{ + \frac{1}{2!} + \frac{1}{2!^2} + + + \frac{1}{2!} + \frac{1}{2!} + \cdots \Big\}$$

- Generator of connected functions:

$$W\{j^*,j\} = \bullet + + \frac{1}{2!^2} + + \frac{1}{2!} + \frac{1}{2!} + \cdots$$

# Lattice Diagrammatics

- 1PI Correlation function:




- Effective Action $\Gamma$ has only 1PI diagrams
- First-order Effective action

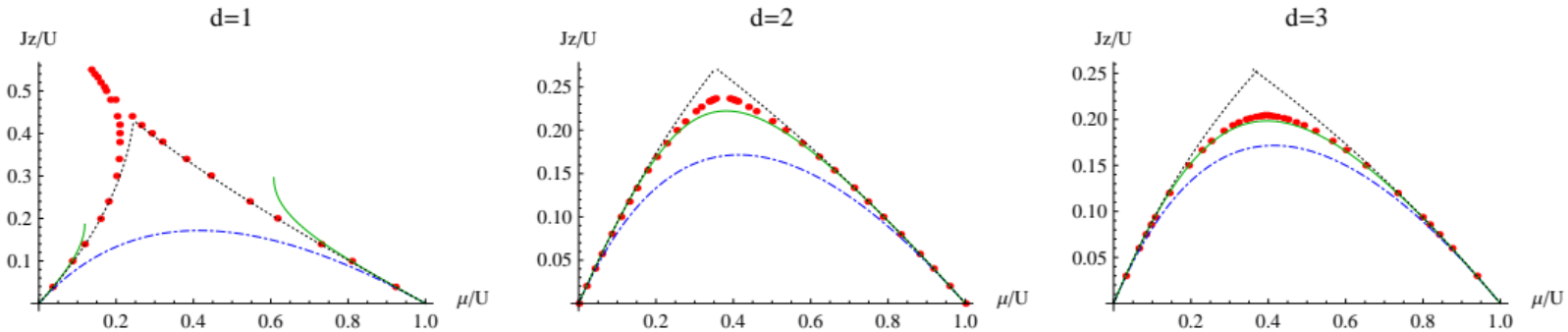
$$\Gamma[\psi,\psi^*] = -\frac{1}{\beta}\sum_{ij} J_{ij}\int_0^\beta d\tau \psi_i^*(\tau)\psi_j(\tau) + \sum_i \Gamma_i[\psi_i^*,\psi_i]$$

- Effective potential (Homogeneus case):

$$\Gamma[\psi,\psi^*]|_{\psi=\text{const.}} = F_0 - \left(\sum_{ij} J_{ij}\right)\psi^*\psi + N\psi^*\psi\int_0^\beta d\tau\int_0^\beta d\tau' \Gamma_i^{(2)}(\tau;\tau') + \cdots$$

# Aplications

- Prediction of Superfluid-Mott Insulator transition from 1989[2]
- The most precise analytical calculations where mean-field [3]
- Lead to errors of up to 16% for 3D lattices
- Effective Action method lead to errors smaller than 3% in 3D[4,5]
- Strong-coupling expansion[6] works better for 1D.




[2] M. P. A. Fisher et al. Phys. Rev. B, 40:546, 1989.

[3] B. Capogrosso-Sansone, et al. Phys. Rev. B, 75:134302, 2007.

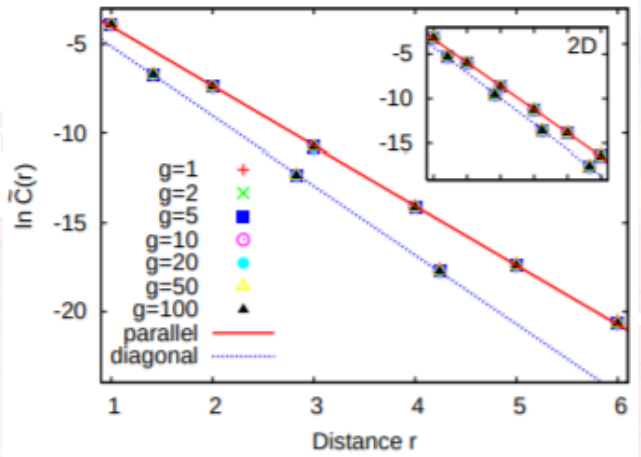
[4] F. E. A. dos Santos and A. Pelster. Phys. Rev. A, 79:013614, 2009.

[5] B. Bradlyn, F. E. A. dos Santos, and A. Pelster. Phys. Rev. A, 79:013615, 2009.

[6] J. K. Freericks and H. Monien. Phys. Rev. B, 53:2691, 1996.

# Aplications

- Systematic higher-order results where obtained by the group of Martin Holthaus[7,8] .




Scaling properties from Ref. 7

- For triangular and kagomé optical lattices considered in Ref. [9]
- High-order symbolic expansion for the Bose-Hubbard model[10] .

[7] N. Teichmann et al. Phys. Rev. B, 79:100503, 2009.

[8] N. Teichmann and D. Hinrichs. Eur. Phys. J. B, 71:219, 2009.

[9] N. Teichmann et al. Europhys. Lett., 91, 10004, 2010.

[10] Tao Wang et al. Phys. Rev. B 98, 245107

# Aplications

- In Refs. [11,12] , Effective action approach with Keldysh formalism was applied to out-of-equilibrium conditions.
- Ref. [13] considered spin-1 bosons in a cubic optical lattices with external magnetic field.
- Ref.[14] the method was used to describe photons evolving in a cubic lattice of cavities through the Jaynes-Cummings-Hubbard model.

[11] T. D. Grass, F. E. A. dos Santos, and A. Pelster. Laser Physics, 21:1459, 2011

[12] T. D. Grass, F. E. A. dos Santos, and A. Pelster. Laser Physics, 21:1459, 2011

[13] Mohamed Mobarak and Axel Pelster. Laser Phys. Lett. 10, 115501, 2013.

[14] Christian Nietner and Axel Pelster. Phys. Rev. A 85, 043831, 2012

# Aplications

- Ref.[15] considered interacting bosons in optical lattice with periodic modulation of the *s*-wave scattering length By combining Effective Action with Floquet theory.
- Quantum phase transitions of ultracold Bose systems in nonrectangular optical lattices[16]
- Tunable anisotropic superfluidity in an optical kagome superlattice[17]

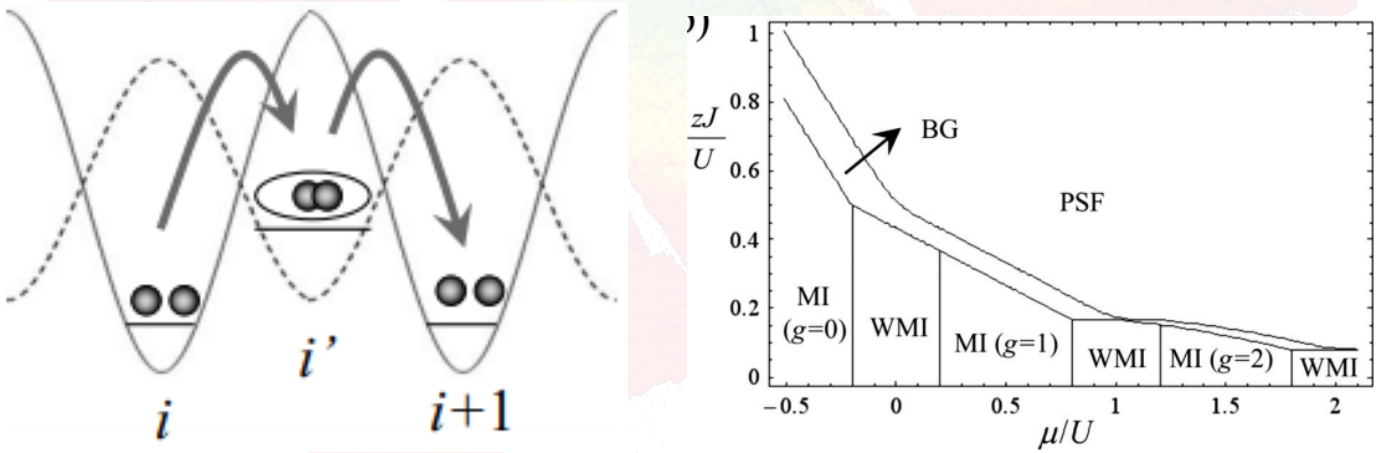
[15] Tao Wang et al. Phys. Rev. A 90, 013633, 2014

[16] Zhi Lin, Jun Zhang, and Ying Jiang, Phys. Rev. A 85, 023619,2012

[17] Xue-Feng Zhang et al. Phys. Rev. B 92, 014512, 2015

# Pair tunneling of bosonic atoms in an optical lattice

- Our theory was applied in Refs. [18,19] to obtain the phase boundary separating the Mott insulator phase from the so called pair superuid phase (PSF) which is induced by excited molecules loaded into the optical lattice.



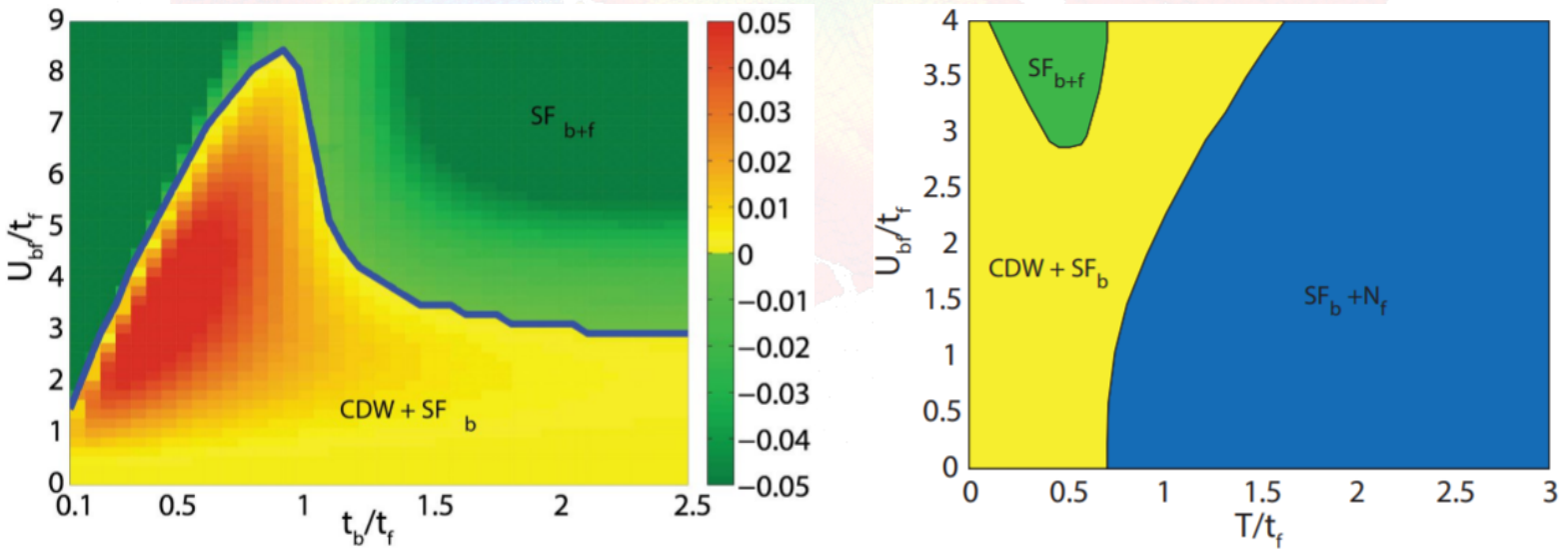

From Ref. 16

[18] Wang Yue-Ming and Liang Jiu-Qing, Chinese Phys. B 21 060305, 2012

[19] Xiang-Fa Zhou et al. Phys. Rev. A 80, 013605, 2009

# Aplications

- In Ref.[20], calculated the phase diagram of attractive atoms with three-body constraint in optical lattices. In particular, they found a dimer superuid phase (DSF) in addition to MI and SF.
- In Ref.[21] the method was used to study Bose-Fermi mixtures and obtain the phase diagram



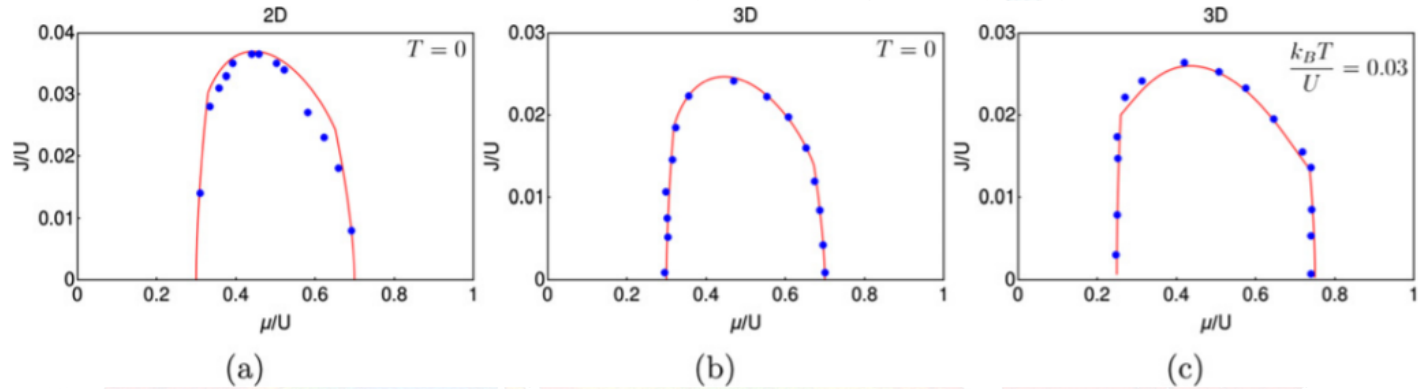

From Ref. 18

[20] Yu-Wen Lee and Min-Fong Yang. Phys. Rev. A 81, 061604(R) , 2010

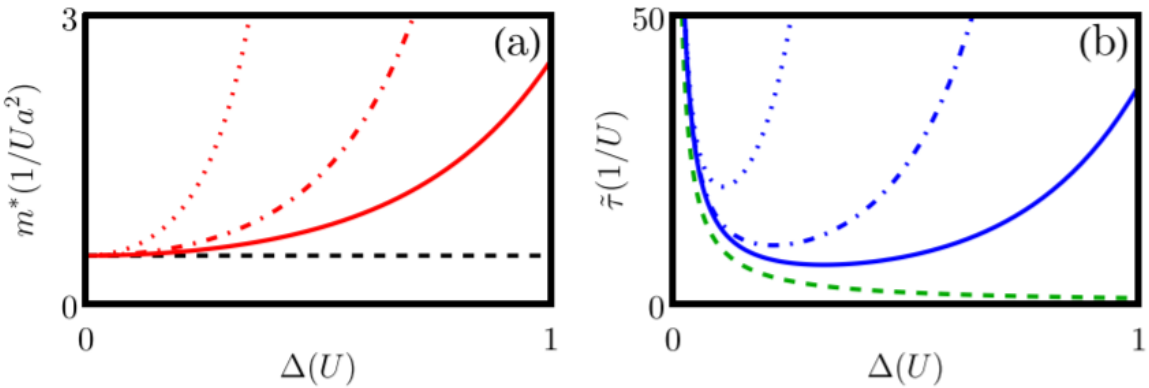
[21] Marin Bukov and Lode Pollet. Phys. Rev. B 89, 094502, 2014

# Aplications

- Quantum phase diagram with disorder obtained in [22]




- Damped-localized excitations were found in [23]




[22] New J. Phys. 23 083007, 2021

[23] New J. Phys. 25 063015, 2023