JOINT RESEARCH CENTRE

Scientific and Technical Lectures

ADDENDUM

Avis/A/39bis/94

We inform you that the Seminar

"An introduction to Synergetics by the example of the laser"
to be held in the Amphitheatre, building 36,
in the afternoon of 15th September 1994 (14.00-18.00)
and in the morning of 16th September 1994 (9.00-11.00)

will be given by Prof. Axel PELSTER
University of Stuttgart
Institute of Theoretical Physics and Synergetics

Co-ordinator:
J.M. Zaldívar
Safety Technology Institute
Tel. 9202 - fax 5024

EUR/IS/103/94 E

INSTITUTE FOR
THEORETICAL PHYSICS AND SYNERGETICS
Professor Dr. Dr. h.c.mult. H.Haken

UNIVERSITY OF
STUTTGART

Pfaffenwaldring 57/4
70550 Stuttgart (Germany)
Tel. 0049/711/685-4990/4989
Telex 7255445 univ d
Telefax 0049/711/685-4909
E-Mail: Haken@theo.physik.uni-stuttgart.de

Axel Pelster

Tel.: 0049/711/685- 4985
E-Mail: Lenin@theo.physik.uni-stuttgart.

Ispra, September 15/16, 1994

An Introduction to Synergetics by the Example of the Laser

References

1. Popularscientific References:

[1] H. Haken: The Science of Structure – Synergetics; Van Nostrand Reinhold Company, 1984

[2] H. Haken, M. Haken-Krell: Entstehung von biologischer Information und Ordnung; Wissenschaftliche Buchgesellschaft, 1989

[3] H. Haken, A. Wunderlin: Die Selbststrukturierung der Materie – Synergetik in der unbelebten Welt; Vieweg, 1991

[4] H. Haken, M. Haken-Krell: Erfolgsgeheimnisse der Wahrnehmung – Synergetik als Schlüssel zum Gehirn; Deutsche Verlags–Anstalt, 1992

2. Scientific References:

[1] H. Haken: Synergetics – An Introduction, Nonequilibrium Phase Transitions and Self-Organization in Physics, Chemistry and Biology; Third Revised and Enlarged Edition, Springer, 1983

[2] H. Haken: Advanced Synergetics – An Instability Hierarchies of Self-Organizing Systems and Devices; Third Printing, Springer, 1993

[3] H. Haken: Information and Self-Organization – A Macroscopic Approach to Complex Systems; Springer, 1983

[4] H. Risken: Fokker-Planck Equation – Methods of Solution and Applications; Second Edition, Springer, 1989

[5] H. Haken: Synergetic Computers and Cognition – A Top-Down Approach to Neural Nets; Springer, 1991

1. The Interdisciplinary Field of Synergetics:

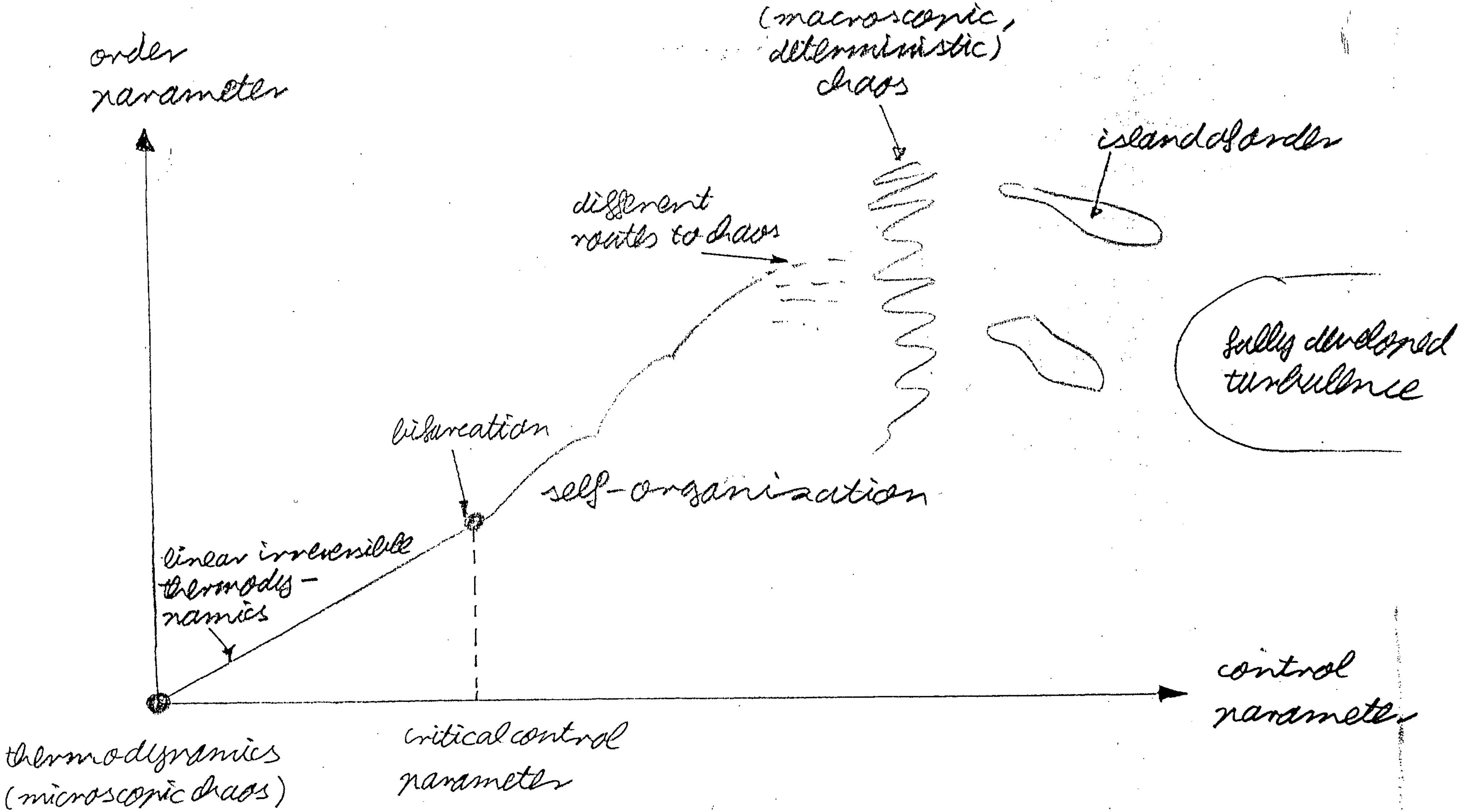
1.1. Slides:

In nature we observe various types of ordered structures on different time and length scales:

- galaxy
- corona
- surface of the venus
- clouds above africa
- cloud hexagons
- cloud streets
- interior of the earth
- continental drift
- magnetic field of the earth
- laminar and turbulent flow
- sand dune
- mussel shell
- fern leaf
- zebra crossing
- EEG of α - waves
- EEG of an epileptic fit

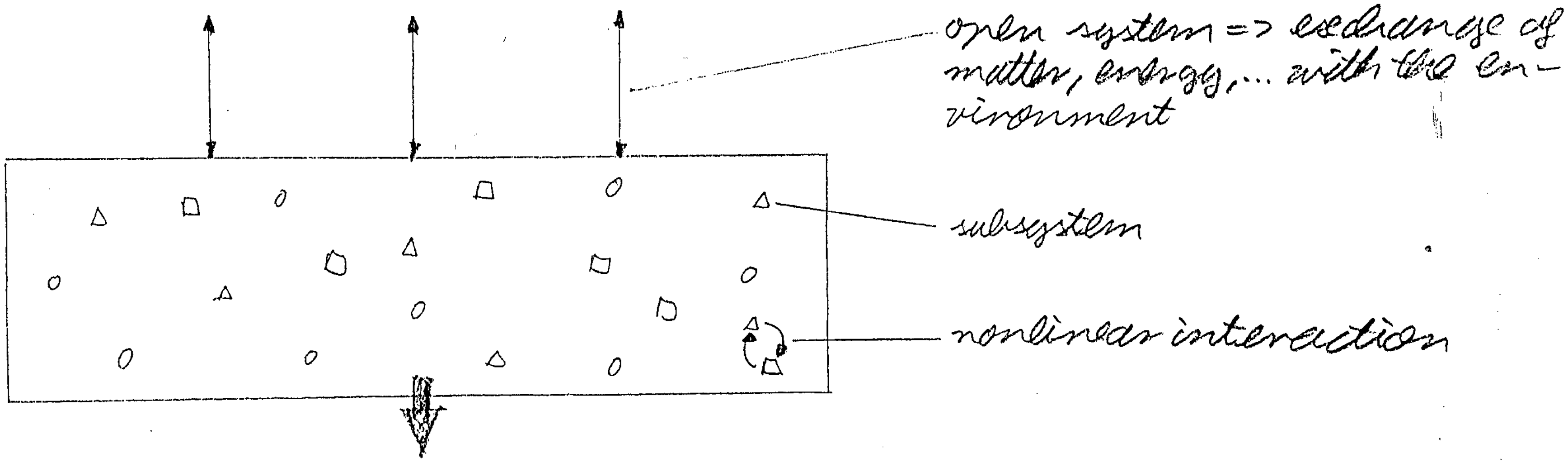
Is it possible to describe the emergence of ordered structures from a unifying point of view?

1.2. Overview:



order parameter
(macroscopic, deterministic) chaos
islands of order
different routes to chaos
fully developed turbulence
bifurcation
self-organization
linear irreversible thermodynamics
control parameter
thermodynamics (microscopic chaos)
critical control parameter

1.3. Synergetics:



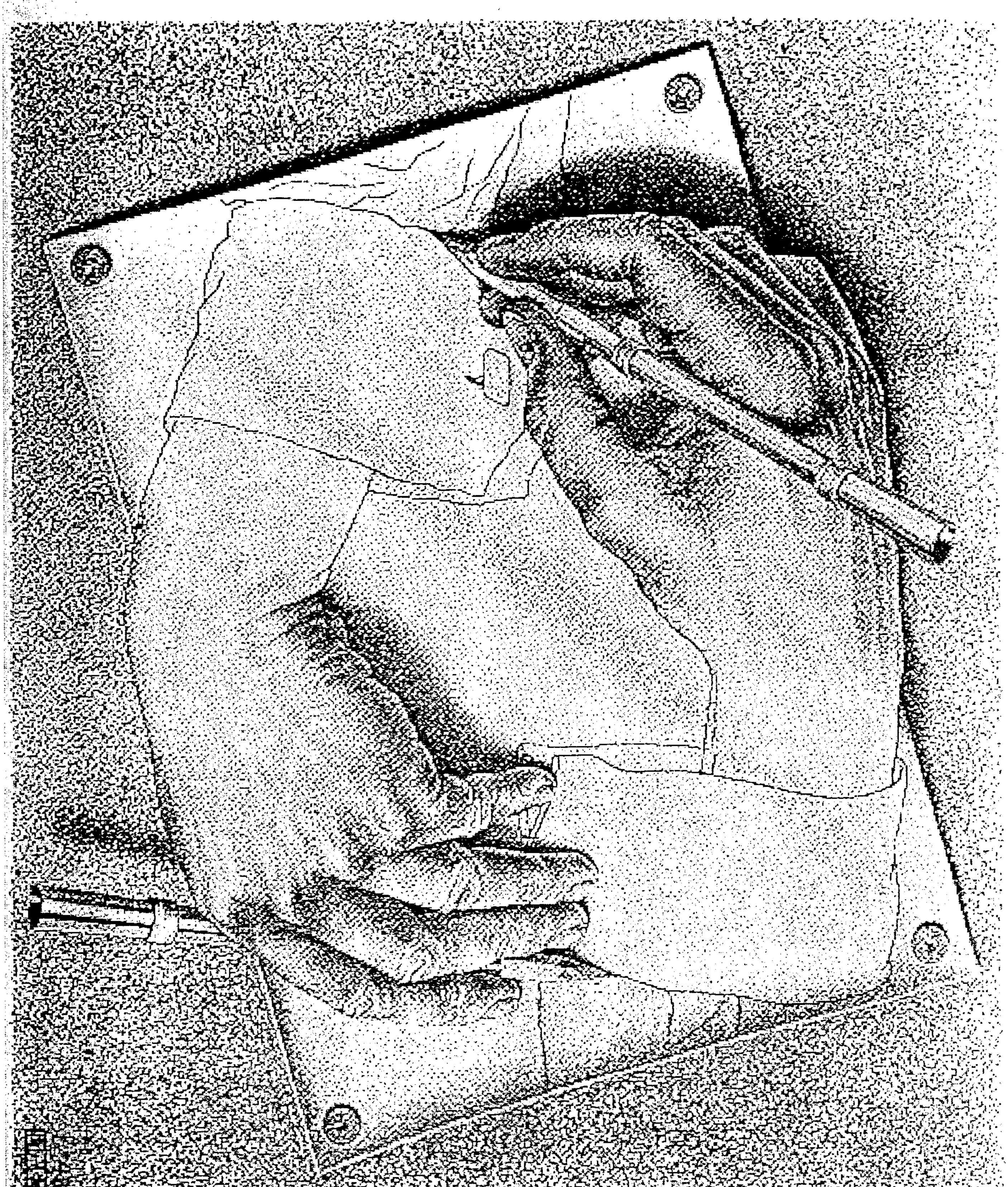
Spatial, temporal, spatio-temporal or functional ordered structures may spontaneously emerge on macroscopic scales.

→ claim: a theory of self-organization (Hermann Haken, 1969)

→ a mathematical structure theory irrespective of the special nature of the subsystems and their nonlinear interactions

→ result of a systematic self-consistency procedure: the slow variables enslave the fast variables and reversely the fast variables have an influence on the evolution of the slow variables (compare the lithography of M.C. Escher from 1946 on the next page and subsection 6.5.)

→ restriction: local analysis in the vicinity of a bifurcation point



2. The Laser as a Synergetic System:

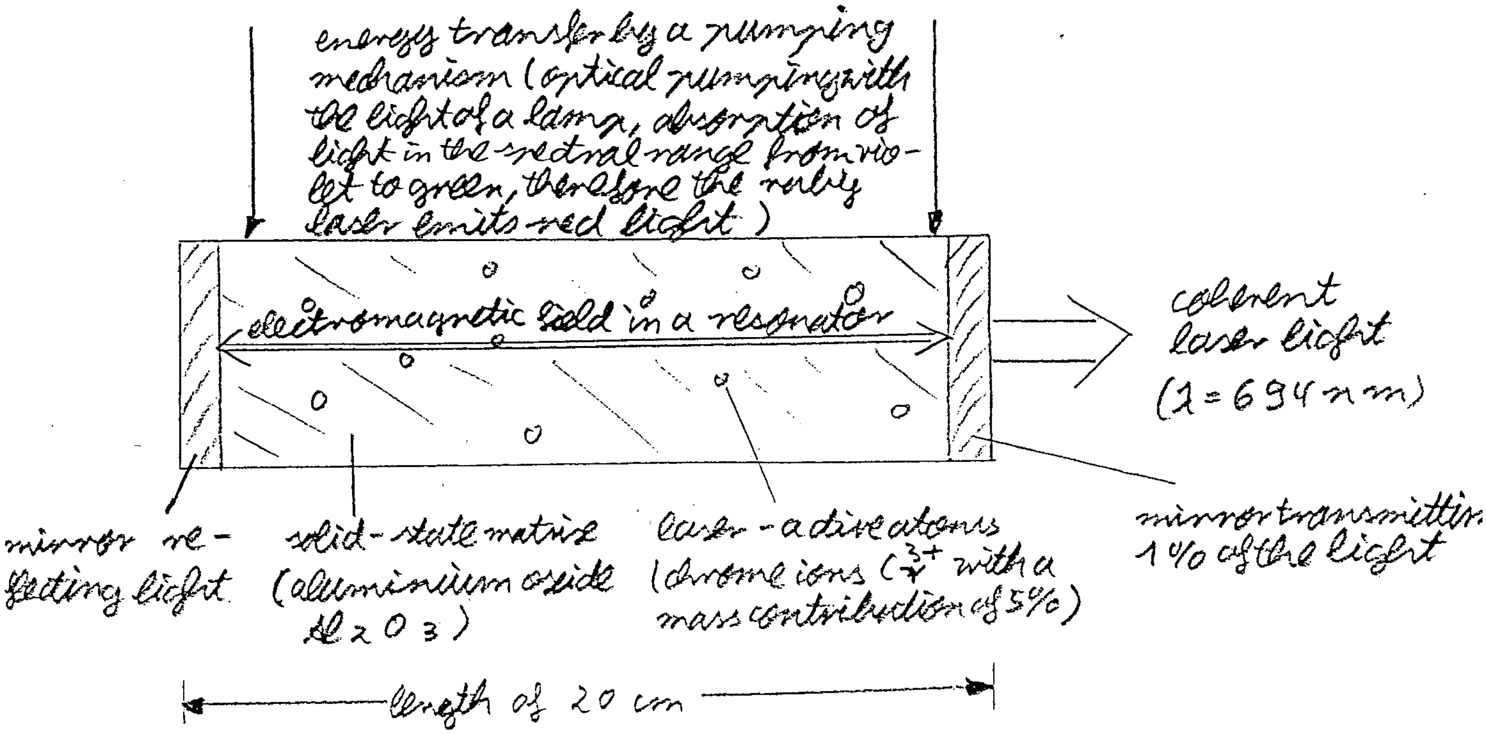
2.1. The Properties of the Laser Light:

"Laser" is an artifical word. It represents an abbreviation for the expression "Light Amplification by Stimulated Emission of Radiation". The laser as a technical device is a self-sustained oscillator which amplifies monochromatic electromagnetic waves in the infrared, the optical and the ultraviolet range. In comparison with the light of an ordinary lamp, the laser light possesses various exceptional properties. Because of them the laser represents an indispensible instrument for experiments and for technical applications:

- The power of a laser is not very high and varies typically between 1 mW and 100 W.
- The laser light is highly directional so that it can be easily focussed.
- Laser light can be generated in form of ultrashort pulses with a duration of 1 fs.
- The laser light has very high intensities. Ultrashort pulses obtain intensities up to $10^{18}\ \frac{W}{cm^2}$.
- The laser light has a relative linewidth up to 10^{-15} so that it is very monochromatic.
- The coherence length of a laser is with 10^{8} m much langer than the coherence length of a lamp with 1 m.

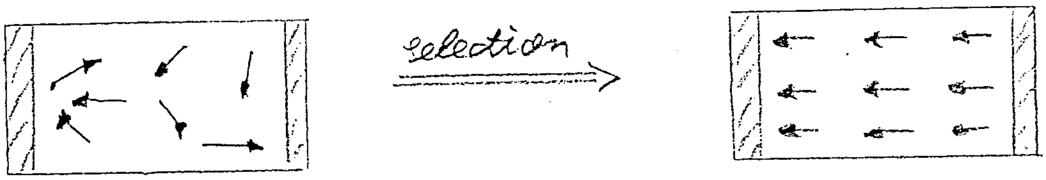
2.2. The Experimental Set-Up for a Laser:

We study the ruby laser as an example of a solid-state laser. It has been invented in 1960, represents a three-level laser and emits red light:



2.3. Mode Selection by the Mirrors:

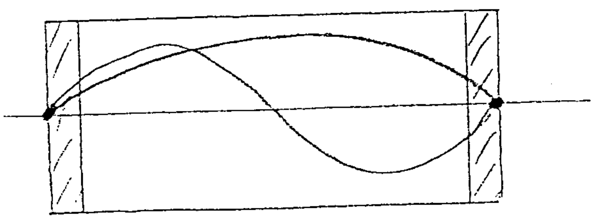
- Due to the arrangement of the mirrors only light which is emitted very close to the axis remains in the resonator long enough. ⇒ Mode selection with respect to the lifetime.



- Because of interference effects only such axial modes can survive in the resonator which vanish at both mirrors.
  ⇒ Mode selection with respect to the wave length.

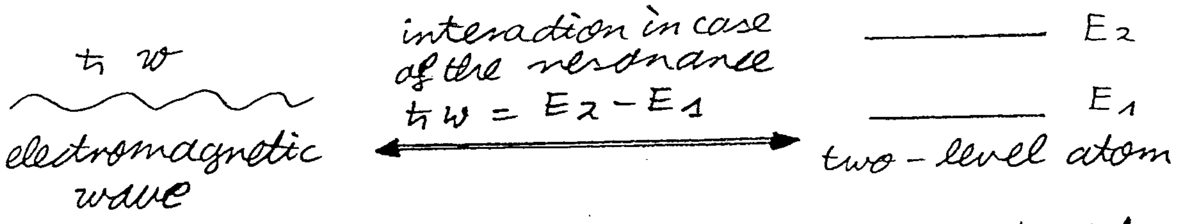
$$\lambda_n = n \cdot \frac{\pi}{L}$$

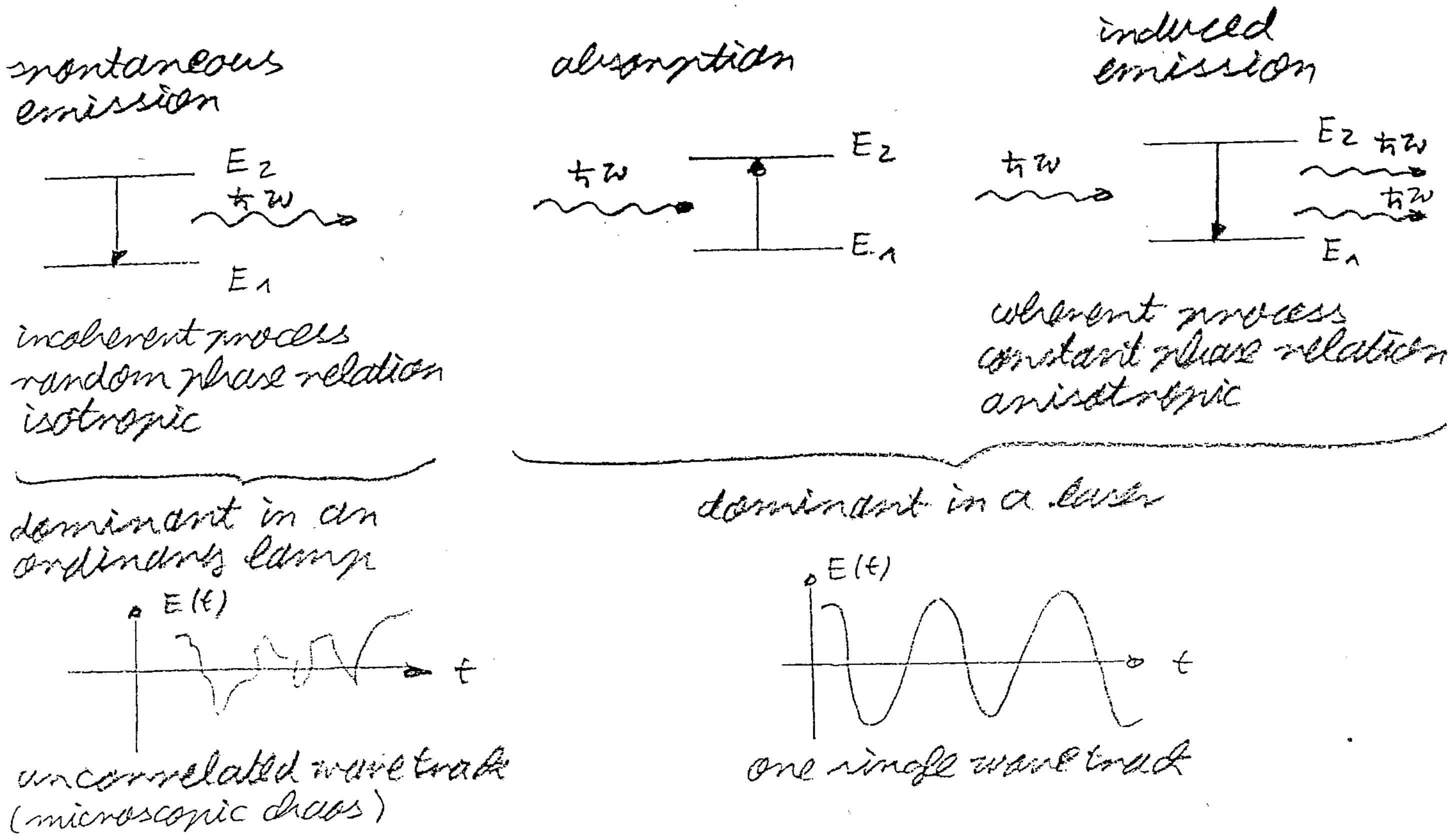
allowed wave lengths

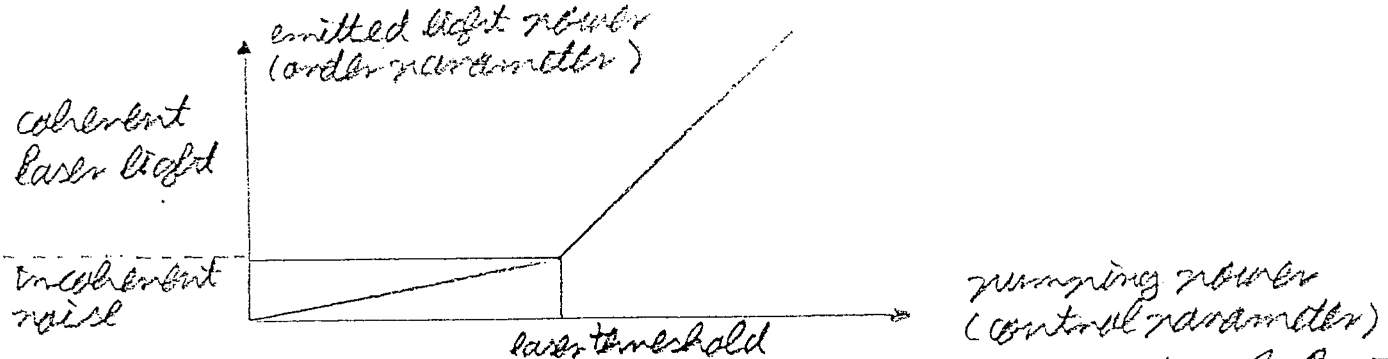


2.4. The Laser Principle:

The interaction of light and matter consists of three elementary processes. They were originally postulated by Einstein in a phenomenological theory to explain the Planck radiation. Later on it became possible to derive them from the first principles of quantum mechanics. For the purpose of illustration we regard a two-level atom which is in resonance with one electromagnetic wave:







In the laser the state of order changes at the laser threshold. This indicates that the laser activity is the result of a self-organization process. Regarding applications it is important to investigate how the laser threshold and the output power as a function of the pumping power depend on the respective laser parameters.

2.5. The Laser Theory:

The "quantum mechanical equations" are obtained by a full quantum mechanical treatment of light and matter. While doing so both light and matter are coupled to an environment so that dissipation and fluctuation effects are induced.	→ line width → intensity fluctuations ⋮

↓ quantum mechanical average concerning light properties

The "semiclassical equations" are derived considering light classically and matter quantum mechanically.	→ frequency shift → ultrashort pulses ⋮

↓ Neglect of all phase relations

The "rate equations" discuss the gains and losses of the photon numbers and the occupation numbers of atomic levels in form of a phenomenological theory.	→ threshold condition → output power as a function of pumping ⋮

progressing simplification ↓

in addition tractable problems ↑

3. The Rate Equations:

3.1. The Derivation of the Rate Equations:

- Assumptions of a phenomenological laser theory:

field	matter
- In the single-mode laser only one mode survives the competition between the different electromagnetic modes. - In a full quantum mechanical treatment one introduces the photon number $n(t)$ whose possible values are positive integers. - In a phenomenological theory one can assume that the photon number $n(t)$ can take positive real values	- The atoms are modelled by two levels: $E_2, N_2(t)$ $E_1, N_1(t)$ - In a phenomenological theory one can assume that the numbers $N_1(t), N_2(t)$ of atoms which occupy the states E_1, E_2 take positive real values.

- The field equation summarizes the gains and losses for the photon number $n(t)$. By neglecting the spontaneous emission, one obtains:

$$\dot{n}(t) = \underbrace{-2\varkappa n(t)}_{\text{resonator losses}} + \underbrace{W_{12} N_2(t) n(t)}_{\text{induced emission}} - \underbrace{W_{21} N_1(t) n(t)}_{\text{absorption}}$$

- The Einstein-coefficients W_{12}, W_{21} characterise the strength of the nonlinear interaction between light and matter. In the phenomenological theory of Einstein for the Planck radiation or in a proper quantum mechanical treatment one can derive the equality of both Einstein-coefficients:

$$W_{12} = W_{21} = W$$

- The inversion is defined as the difference between the occupation numbers in the higher and in the lower state:

$$D(t) = N_2(t) - N_1(t)$$

- With this the field equation for the photon number $n(t)$ reduces to

$$\dot{n}(t) = -2\varkappa n(t) + W D(t) n(t)$$

- In order to obtain a closed system of equations it becomes necessary to regard the evolution of the inversion $D(t)$. To this end one has to take into account the gains and losses for the occupation numbers $N_1(t), N_2(t)$:

$$\dot{N}_2(t) = w_{21} N_1(t) - w_{12} N_2(t) + W_{21} N_1(t) n(t) - W_{12} N_2(t) n(t)$$

$$\dot{N}_1(t) = \underbrace{-w_{21} N_1(t)}_{\substack{\text{pumping}\\ \text{process}}} + \underbrace{w_{12} N_2(t)}_{\substack{\text{nonradiative}\\ \text{transition}}} - \underbrace{W_{21} N_1(t) n(t)}_{\text{absorption}} + \underbrace{W_{12} N_2(t) n(t)}_{\substack{\text{induced}\\ \text{emission}}}$$

- we remark that the nonradiative transitions are due to interactions between the laser-active atoms and the solid-state matrix. The energy $E_2 - E_1$ which is gained by a nonradiative transition leads to oscillations of the whole solid-state matrix.
- As a result of the different processes taken into account for the gains and losses of the occupation numbers $N_1(t), N_2(t)$ the total number $N(t)$ of laser-active atoms is conserved:

$$N(t) = N_2(t) + N_1(t), \quad \dot{N}(t) = 0 \quad \Rightarrow \quad N(t) = N.$$

- The difference of both evolution equations for the occupation numbers $N_1(t), N_2(t)$ leads to one for the inversion $D(t)$:

$$\dot{D}(t) = 2\, w_{21} N_1(t) - 2\, w_{12} N_2(t) + 2\, W_{21} N_1(t)\, n(t) - 2\, W_{12} N_2(t)\, n(t)$$

$$= w_{21} \{ N - D(t) \} - w_{12} \{ N + D(t) \} - 2\, W\, D(t)\, n(t)$$

$$\Rightarrow \quad \dot{D}(t) = (w_{21} - w_{12})\, N - (w_{21} + w_{12})\, D(t) - 2\, W\, D(t)\, n(t)$$

- If the nonlinear interaction between light and matter is neglected according to $W = 0$ then the equilibrium between the pumping and the nonradiative transitions leads to the unsaturated inversion:

$$D_0 = N \cdot \frac{w_{21} - w_{12}}{w_{21} + w_{12}}$$

In order to guarantee that the unsaturated inversion D_0 takes only positive values, the pumping parameter w_{21} has to be larger than the parameter w_{12} for the nonradiative transitions.
- The inversion $D(t)$ tends to the unsaturated inversion D_0. This relaxation takes place on a time scale which is defined by the reciprocal of the relaxation parameter

$$\gamma = w_{21} + w_{12}$$

- with this the matter equation for the inversion $D(t)$ reduces to

$$\dot{D}(t) = \gamma \{ D_0 - D(t) \} - 2\, W\, D(t)\, n(t)$$

- The field equation and the matter equation represent the rate equations of a phenomenological laser theory.
- without the pumping mechanism, that is in the case $D_0 = 0$, the rate equations have the form of the Lotka-Volterra model which was originally designed to explain temporal oscillations of fish populations. To this end one has to identify the photon number $n(t)$ with the number of predator fishes and the inversion $D(t)$ with the number of prey fishes.
- It is not possible to obtain the complete analytic solution of the rate equations. Therefore one performs approximations in such a way that one can extract at least the most important information of the dynamics.

3.2. Elementary Properties of Dynamical Systems:

- We consider a dynamical system which is defined by two first-order ordinary differential equations:

$$\dot{q}_1(t) = F_1(q_1(t), q_2(t))$$
$$\dot{q}_2(t) = F_2(q_1(t), q_2(t))$$

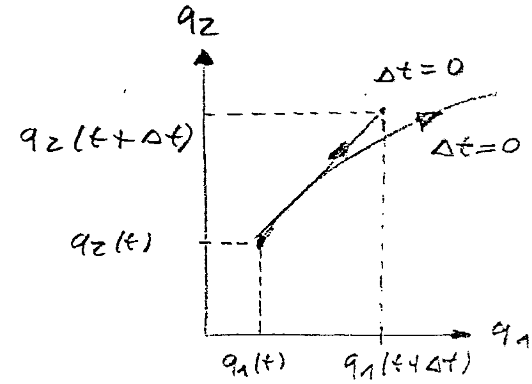
As the nonlinear functions F_1, F_2 do not explicitly depend on the time coordinate t we confine our analysis to a so-called autonomous system.

- Recalling the definition of the total derivative

$$\dot{q}_1(t) = \frac{dq_1(t)}{dt} = \lim_{\Delta t \to 0} \frac{q_1(t+\Delta t) - q_1(t)}{\Delta t}$$

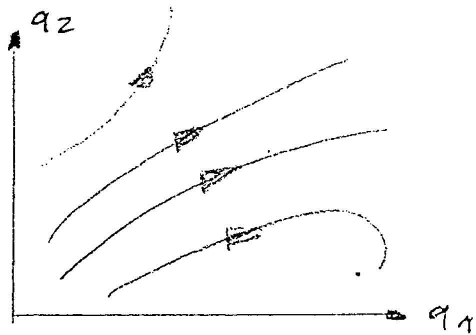
we obtain the Euler procedure to integrate the set of ordinary differential equations numerically:

$$q_1(t+\Delta t) = q_1(t) + \Delta t \cdot F_1(q_1(t), q_2(t))$$
$$q_2(t+\Delta t) = q_2(t) + \Delta t \cdot F_2(q_1(t), q_2(t))$$



If q_1, q_2 are given at time t this formula allows to calculate their values at the later time $t+\Delta t$. Repeating this procedure we find in the limit $\Delta t \to 0$ a trajectory in the q_1, q_2-plane.

- If we choose different initial values for the Euler procedure we obtain a field of trajectories which can be interpreted as streamlines in a fluid. Thereby the trajectories do not cross as this would contradict the uniqueness of the trajectories.



- The simplest solution of the set of ordinary differential equations is provided by the case that a whole trajectory consists of one singular point q_1^0, q_2^0. This stationary solution is defined by the condition

$$F_1(q_1^0, q_2^0) = F_2(q_1^0, q_2^0) = 0$$

- In order to determine if the singular point q_1^0, q_2^0 is stable or unstable we have to take into account trajectories in its neighborhood:
  - A singular point q_1^0, q_2^0 is asymptotically stable if all trajectories starting sufficiently near it tend to it asymptotically for $t \to +\infty$.
  - A singular point q_1^0, q_2^0 is asymptotically unstable if some trajectories which are sufficiently close to it tend for $t \to +\infty$ asymptotically away from it.

- The behaviour of the trajectories near a singular point q_1^0, q_2^0 can be classified. To this end we introduce deviations $\tilde{q}_1(t), \tilde{q}_2(t)$ from this singular point q_1^0, q_2^0 according to

$$q_1(t) = q_1^0 + \tilde{q}_1(t), \quad \tilde{q}_2(t) = q_2^0 + \tilde{q}_2(t)$$

Then we expand the nonlinear functions F_1, F_2 up to the first order in these deviations $\tilde{q}_1(t), \tilde{q}_2(t)$:

$$\dot{\tilde{q}}_1(t) = F_1(q_1^0+\tilde{q}_1(t), q_2^0+\tilde{q}_2(t)) = F_1(q_1^0, q_2^0) + \frac{\partial F_1(q_1^0,q_2^0)}{\partial q_1^0}\tilde{q}_1(t) + \frac{\partial F_2(q_1^0,q_2^0)}{\partial q_2^0}\tilde{q}_2(t)$$

$$\dot{\tilde{q}}_2(t) = F_2(q_1^0+\tilde{q}_1(t), q_2^0+\tilde{q}_2(t)) = F_2(q_1^0, q_2^0) + \frac{\partial F_2(q_1^0,q_2^0)}{\partial q_1^0}\tilde{q}_1(t) + \frac{\partial F_2(q_1^0,q_2^0)}{\partial q_2^0}\tilde{q}_2(t).$$

$$\Longrightarrow \frac{d}{dt}\begin{pmatrix}\tilde{q}_1(t)\\ \tilde{q}_2(t)\end{pmatrix} = \mathcal{L}\begin{pmatrix}\tilde{q}_1(t)\\ \tilde{q}_2(t)\end{pmatrix}, \quad \mathcal{L} = \begin{pmatrix}\mathcal{L}_{11} & \mathcal{L}_{12}\\ \mathcal{L}_{21} & \mathcal{L}_{22}\end{pmatrix}, \quad \mathcal{L}_{ij} := \left.\frac{\partial F_i(q_1,q_2)}{\partial q_j}\right|_{\substack{q_1=q_1^0\\ q_2=q_2^0}}$$

- By the aid of a subsequent linear coordinate transformation which represents a combination of a rotation with a deformatio[n]

$$\begin{pmatrix}\tilde{q}_1(t)\\ \tilde{q}_2(t)\end{pmatrix} = T\begin{pmatrix}\xi_1(t)\\ \xi_2(t)\end{pmatrix}, \quad T = \begin{pmatrix}T_{11} & T_{12}\\ T_{21} & T_{22}\end{pmatrix}$$

it is often possible to diagonalise this linearised ordinary differential equation. Then the corresponding solution can be immediately written down:

$$\dot{\xi}_1(t) = \lambda_1 \xi_1(t) \quad \Longrightarrow \quad \xi_1(t) = \xi_1(0)\cdot e^{\lambda_1 t}$$

$$\dot{\xi}_2(t) = \lambda_2 \xi_2(t) \quad \Longrightarrow \quad \xi_2(t) = \xi_2(0)\cdot e^{\lambda_2 t}$$

- The technical details of this diagonalisation procedure shall be further developed in subsection 5.2. At this stage it is only necessary to remark that λ_1, λ_2 are the so-called eigenvalues of the matrix $\mathcal{L}$ and that they are determined as the zeros of a polynomial of second order in λ:

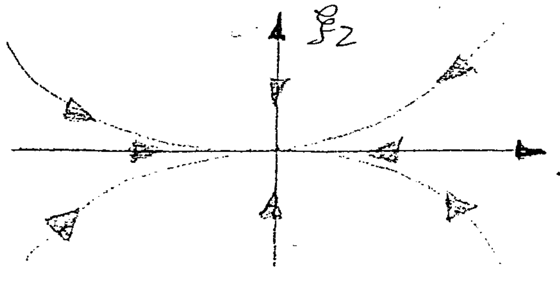
$$\det(\mathcal{L} - \lambda I) = \det\begin{pmatrix}\mathcal{L}_{11}-\lambda & \mathcal{L}_{12}\\ \mathcal{L}_{21} & \mathcal{L}_{22}-\lambda\end{pmatrix} = (\mathcal{L}_{11}-\lambda)(\mathcal{L}_{22}-\lambda) - \mathcal{L}_{12}\mathcal{L}_{21} =$$

In this way one obtains the characteristic equation of the matrix $\mathcal{L}$

$$\lambda^2 - \underbrace{(\mathcal{L}_{11}+\mathcal{L}_{22})}_{=\mathrm{Tr}\,\mathcal{L}}\cdot\lambda + \underbrace{(\mathcal{L}_{11}\mathcal{L}_{22} - \mathcal{L}_{12}\mathcal{L}_{21})}_{=\det\mathcal{L}} = 0$$

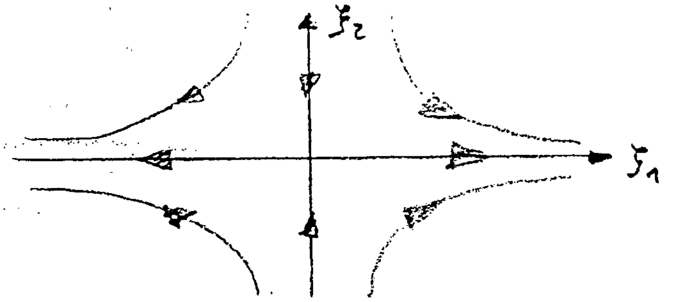
- According to the eigenvalues λ_1, λ_2 it is possible to distingui[sh] the following cases:

λ_1, λ_2 real, $\lambda_1 \gtrless 0$, $\lambda_2 \gtrless 0$



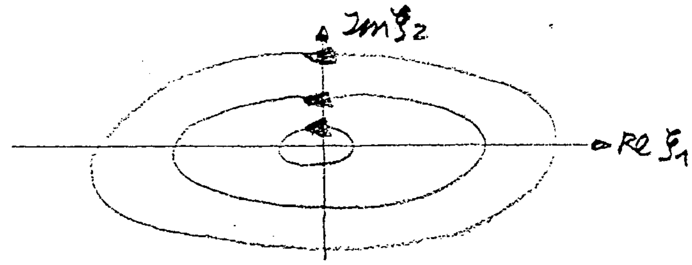
stable (unstable) node

λ_1, λ_2 real, $\lambda_1 \gtrless 0, \lambda_2 \lessgtr 0$



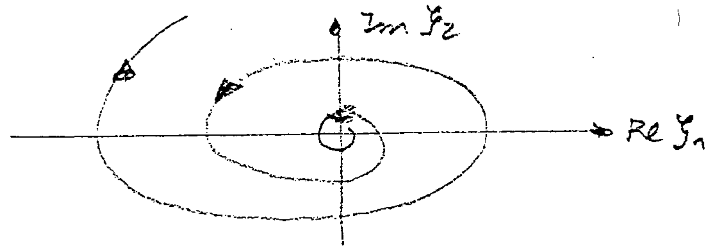
unstable saddle

$\lambda_1 = -i\omega, \; \lambda_2 = +i\omega = \lambda_1^*$



center

$\lambda_1 = -\gamma \mp i\omega, \; \lambda_2 = -\gamma \pm i\omega = \lambda_1^*$



stable (unstable focus

- Both the stationary solutions and their corresponding eigenvalues depend on the control parameters of the system. If these control parameters are increased two things can happen:
  - The stability of a stationary solution changes.
  - New stationary solutions emerge.

3.3. The Linear Stability Analysis of the Rate Equations:

- The rate equations of the laser

$$\dot{n}(t) = F_1(n(t), D(t)) = -2\kappa n(t) + W D(t) n(t)$$

$$\dot{D}(t) = F_2(n(t), D(t)) = \gamma_0 \{D_0 - D(t)\} - 2 W D(t) n(t)$$

possess two stationary solutions:

$n_1^0 = 0, \qquad D_1^0 = D_0, \qquad D_0 \geq 0 \qquad \hat{=}$ lamp light

$n_2^0 = \frac{\gamma}{4\kappa}\left(D_0 - \frac{2\kappa}{W}\right), \quad D_2^0 = \frac{2\kappa}{W}, \qquad D_0 \geq D_0^I = \frac{2\kappa}{W} \qquad \hat{=}$ laser light

- In order to perform a linear stability analysis around both stationary solutions one needs the Jacobian matrise:

$$\mathcal{L}(n, D) = \begin{pmatrix} \frac{\partial F_1(n,D)}{\partial n} & \frac{\partial F_1(n,D)}{\partial D} \\ \frac{\partial F_2(n,D)}{\partial n} & \frac{\partial F_2(n,D)}{\partial D} \end{pmatrix} = \begin{pmatrix} -2\kappa + W D & W n \\ -2 W D & -\gamma - 2 W n \end{pmatrix}$$

- The linear stability analysis of the first stationary solution yields:

$$\mathcal{L}_1 = \mathcal{L}(n_1^0, D_1^0) = \begin{pmatrix} -2\kappa + W D_0 & 0 \\ -2 W D_0 & -\gamma \end{pmatrix}$$

$\Rightarrow \lambda_{11}(D_0) = -2\kappa + W D_0, \quad \lambda_{12}(D_0) = -\gamma$

when the control parameter D_0 is increased the stability of the first stationary solution changes:

$D_0 < D_0^I = \frac{2\kappa}{W}: \quad \lambda_{11}(D_0) < 0, \lambda_{12}(D_0) < 0 \Rightarrow$ stable node

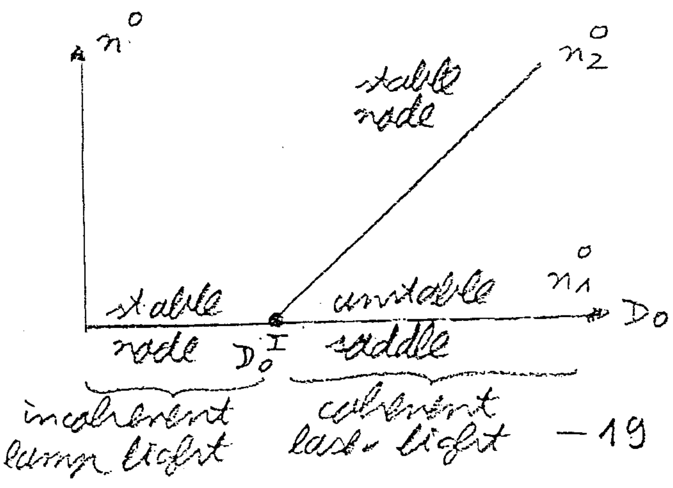
$D_0 > D_0^I = \frac{2\kappa}{W}: \quad \lambda_{21}(D_0) > 0, \lambda_{22}(D_0) < 0 \Rightarrow$ unstable saddle

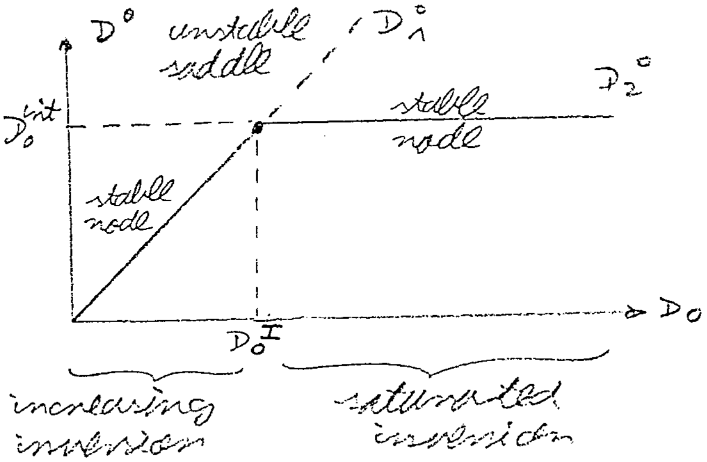
- The linear stability analysis of the second stationary solution yields:

$$\mathcal{L}_2 = \mathcal{L}(n_2^0, D_2^0) = \begin{pmatrix} 0 & \frac{\gamma W}{4\kappa}\left(D_0 - \frac{2\kappa}{W}\right) \\ -4\kappa & -\gamma - \frac{\gamma W}{2\kappa}\left(D_0 - \frac{2\kappa}{W}\right) \end{pmatrix}$$

$$\left.\begin{aligned} &\Rightarrow \lambda_{21}(D_0) = -W\left(D_0 - \frac{2\kappa}{W}\right) + O\left(\left(D_0 - \frac{2\kappa}{W}\right)^2\right) < 0 \\ &\Rightarrow \lambda_{22}(D_0) = -\gamma + \left(W - \frac{\gamma W}{2\kappa}\right)\left(D_0 - \frac{2\kappa}{W}\right) + O\left(\left(D_0 - \frac{2\kappa}{W}\right)^2\right) < 0 \end{aligned}\right\} \text{stable node}$$

- The results of the linear stability analysis can be summarised by the following bifurcation diagrams (compare subsection 2.4):



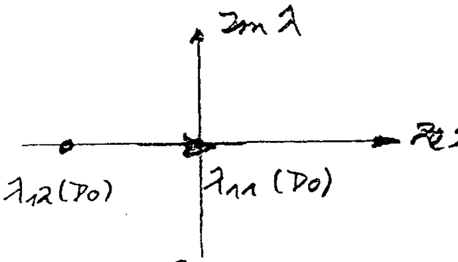


3.4. Adiabatic Elimination:

- From the linear stability analysis of the rate equations one can deduce that the dynamical behaviour of the laser changes qualitatively if the control parameter D_0 reaches the critical value D_0^I. This motivates to perform in the following a nonlinear analysis of the rate equations which is valid in the neighborhood of this critical value:

$$D_0 \approx D_0^I = \frac{2\varkappa}{W}$$

- In this region of the control parameter the eigenvalues of the first stationary solution obey an inequality:



$$|\lambda_{11}(D_0)| \ll |\lambda_{12}(D_0)|$$

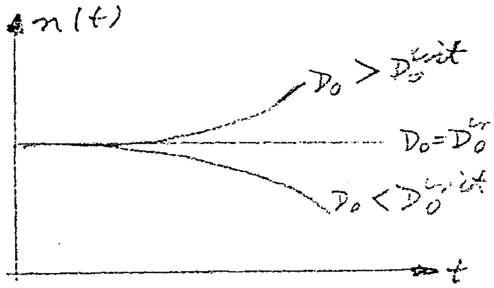
- Each eigenvalue generates a typical time scale:

$$\tau_n = \frac{1}{|\lambda_{11}(D_0)|}, \quad \tau_D = \frac{1}{|\lambda_{12}(D_0)|}$$

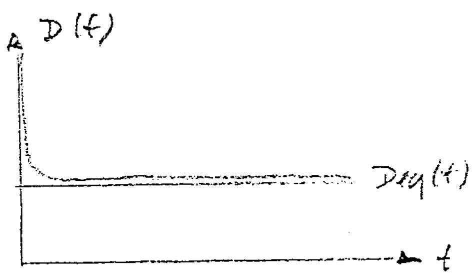
Therefore the inequality of the eigenvalues is equivalent to a time scale hierarchy:

$$\tau_n \gg \tau_D$$

- The photon number $n(t)$ as the unstable mode is a slowly varying quantity. Depending on the value of the control parameter D_0 the photon number $n(t)$ slowly decreases, does not change or slowly increases.



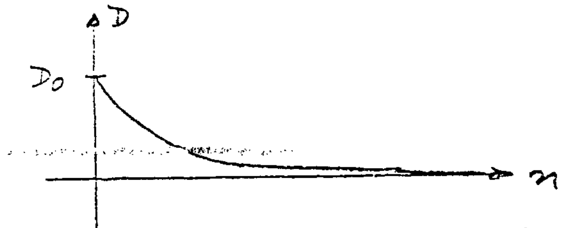
- Contrary to that the inversion $D(t)$ as the stable mode is a fast decreasing quantity. Therefore the inversion $D(t)$ quasiinstantaneously tends to an equilibrium value $D_{eq}(t) = D_{eq}(n(t))$ which is prescribed by the slowly varying quantity, that is the photon number $n(t)$.



- In order to obtain this equilibrium value one applies the approximation of an adiabatic elimination of the fast decreasing quantity, that is the inversion:

$$\dot{D}(t) = \gamma\{D_0 - D(t)\} - 2W D(t)\, n(t) \simeq 0 \Rightarrow D(t) = \frac{D_0}{1 + \frac{2W}{\gamma} \cdot n(t)}$$

- The real, saturated inversion $D(t)$ is smaller than the unsaturated inversion D_0 prescribed by the pumping mechanism.



- Near the instability when the photon number $n(t)$ is small this relation can be further approximated by

$$D(t) = D_0 - \frac{2WD_0}{\gamma}\, n(t) + \mathcal{O}(n(t)^2)$$

- In the language of synergetics one says that the photon number $n(t)$ has enslaved the inversion $D(t)$. This has the consequence that the inversion $D(t)$ is no longer regarded as an independent dynamical variable so that the two-dimensional system effectively reduces to a one-dimensional one.
- Indeed inserting the relation between the inversion $D(t)$ and the photon number $n(t)$ in the original ordinary differential equation of the photon number $n(t)$ one results in:

$$\dot{n}(t) = (WD_0 - 2\varkappa)\, n(t) - \frac{2W^2D_0}{\gamma}\, n(t)^2 + \mathcal{O}(n(t)^3)$$

This is the order parameter equation which correctly describes the activity of the laser near the threshold. It is the normal form of a transcritical bifurcation.
- The order parameter equation can be reformulated by introducing a conservative potential:

$$\dot{n}(t) = -\frac{\partial V(n(t))}{\partial n(t)}, \qquad V(n) = \frac{1}{2}(2\varkappa - WD_0)\, n^2 + \frac{2W^2D_0}{\gamma}\, n^3 + \mathcal{O}(n^4)$$

- Then the order parameter equation can be compared with the evolution equation of a classical particle in a conservative potential under the influence of the friction:

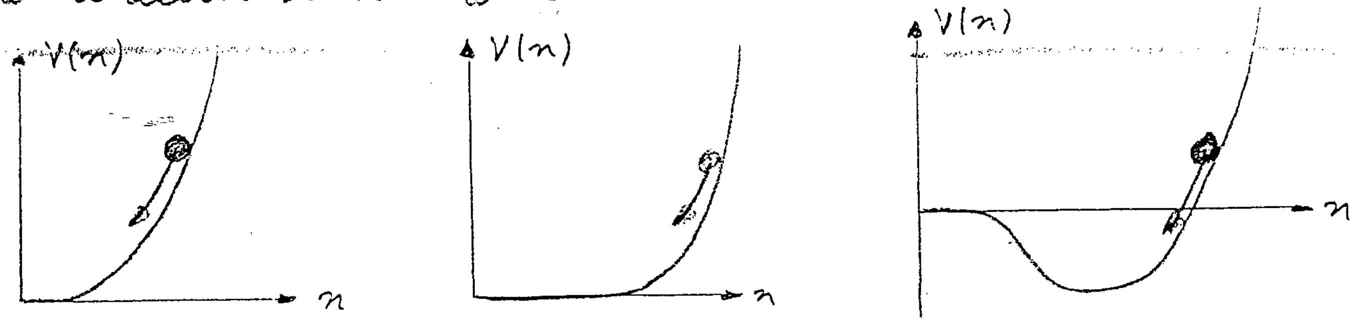
$$\underbrace{m\,\ddot{x}(t)}_{\text{inertial force}} = \underbrace{-\,\alpha\,\dot{x}(t)}_{\text{friction force}} \underbrace{-\,\frac{\partial \mathcal{V}(x(t))}{\partial x(t)}}_{\text{conservative force}}$$

- In the overdamped case it is allowed to neglect the inertial force. By choosing an appropriate time-scale according to $t' = \alpha t$ and $x'(t') = x\left(\frac{t}{\alpha}\right)$ we obtain:

$$\frac{dx'(t')}{dt'} = -\frac{\partial \mathcal{V}(x'(t'))}{\partial x'(t')}$$

This result shows that the order parameter equation can be interpreted as the overdamped motion of a particle in a conservative potential.

- The dependence of the potential shape on the control parameter is illustrated as follows:



$D_0 < D_0^{crit} = \frac{2\varkappa}{W}$ $\qquad$ $D_0 = D_0^{crit} = \frac{2\varkappa}{W}$ $\qquad$ $D_0 > D_0^{crit} = \frac{2\varkappa}{W}$

- The extrema of the potential correspond to the stationary solutions of the order parameter equation:

$$n_1^0 = 0\,,\quad n_2^0 = \frac{\gamma}{2WD_0}\left(D_0 - \frac{2\varkappa}{W}\right) \approx \frac{\gamma}{4\varkappa}\left(D_0 - \frac{2\varkappa}{W}\right)$$

In this way one reobtains the results of the linear stability analysis from the nonlinear analysis.

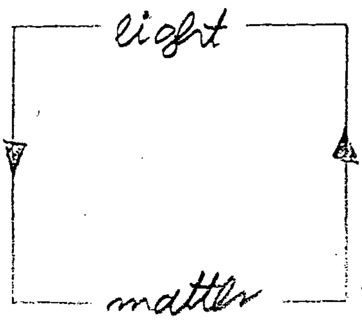
- But in addition it is possible to integrate the order parameter equation analytically by applying the separation method. In this way one obtains an approximation for the time evolution of the photon number $n(t)$ and the inversion $D(t)$ which is valid near the instability.

4. The Semiclassical Laser Theory:

4.1. The Fundamental Notions:

- In the semiclassical laser theory the electromagnetic field is treated classically whereas the ~~matter is treated~~ quantum mechanically.
- The laser activity results from a nonlinear interaction between light and matter:

If the light state is known than the resulting matter state can be calculated by solving the corresponding Schrödinger equation of nonrelativistic quantum mechanics.



If the matter state is known than the resulting light state can be calculated by solving the corresponding Maxwell equations of electrodynamics.

- In both considerations one assumes the following:
  - The laser-active atoms are modelled by a two-level system.
  - The wave functions corresponding to different atoms do not overlap. This has the consequence that the atoms do not interact directly with each other but only interact indirectly via the electromagnetic field.
  - The interaction between light and matter is dominated by the dipole interaction with the following Hamilton operator:

    $$\hat{H} = -\hat{\vec{P}}\,\vec{E}\ , \quad \hat{\vec{P}} = -\,q\cdot\vec{x}$$

    This is the single nonlinearity which appears in the considerations.
  - The spontaneous emission of radiation is neglected.
  - Rotating wave approximation: If ω_λ denotes a typical frequency of the electromagnetic field and $\overline{\omega}_\mu$ a typical atomic frequency than the dipole interaction generates both combination frequencies $\omega_\lambda \pm \overline{\omega}_\mu$. If one performs an average over the period of the smaller frequency $\omega_\lambda - \overline{\omega}_\mu$ it is allowed to neglect the fast oscillating contributions of the higher frequency $\omega_\lambda + \overline{\omega}_\mu$.
  - Slowly varying amplitude approximation: The oscillations of the amplitude of the electromagnetic field have to be slow in comparison with the dominating frequency ω_λ.

4.2. Physical Quantities:

- The electrical field $\vec{E}(\vec{x},t)$ is decomposed in the different resona... eigenmodes $\vec{u}_\lambda(\vec{x})$:

$$\vec{E}(\vec{x},t) = \sum_\lambda i\sqrt{2\hbar\omega_\lambda}\; b_\lambda(t)\,\vec{u}_\lambda(\vec{x}) + c.c.$$

ω_λ: frequency of the resonator mode λ

$b_\lambda(t)$: dimensionless amplitude of the resonator mode λ

- The polarisation $\vec{P}(\vec{x},t)$ is the sum over all atomic contributions:

$$\vec{P}(\vec{x},t) = \sum_\mu \delta(\vec{x}-\vec{x}_\mu)\,\vec{\Theta}_\mu\,\alpha_\mu(t) + c.c.$$

$\vec{x}_\mu$: position of the μ.th laser-active atom in the solid-state matr...

$\vec{\Theta}_\mu$: dipole matrix element for the transition between both level... of the μ.th laser-active atom

$\alpha_\mu(t)$: dimensionless amplitude for the polarisation of the μ.th laser-active atom

- The inversion $\vec{D}(\vec{x},t)$ is also a sum over all atomic contributions:

$$\vec{D}(\vec{x},t) = \sum_\mu \delta(\vec{x}-\vec{x}_\mu)\,d_\mu(t)$$

$d_\mu(t)$: inversion of the μ.th laser-active atom

4.3. Semiclassical Laser Equations:

$$\dot{b}_\lambda(t) = (-i\omega_\lambda - \varkappa_\lambda)\,b_\lambda(t) - i\sum_\mu g^*_{\mu\lambda}\,\alpha_\mu(t) \quad \Big\}\ \text{field equation}$$

$$\left.\begin{aligned}\dot{\alpha}_\mu(t) &= (-i\overline{\omega}_\mu - \gamma_{\perp\mu})\,\alpha_\mu(t) + i\sum_\lambda g_{\mu\lambda}\,b_\lambda(t)\,d_\mu(t)\\ \dot{d}_\mu(t) &= \gamma_{\parallel\mu}\{d_0 - d_\mu(t)\} + 2i\sum_\lambda\{g^*_{\mu\lambda}\,\alpha_\mu(t)\,b^*_\lambda(t) - c.c.\}\end{aligned}\right\}\ \text{matter equations}$$

$\varkappa_\lambda$: resonator losses of the resonator mode λ.

$\overline{\omega}_\mu$: frequency of the μ.th laser-active atom which corresponds according to Bohr to the energy difference between both levels

$\gamma_{\perp\mu}$: transversal relaxation parameter which characterises the width of the frequency of the μ.th laser-active atom

$\gamma_{\parallel\mu}$: longitudinal relaxation parameter which is due to the interaction between the μ.th laser-active atom and the solid-state matrix

d_0: unsaturated inversion generated by the pumping mechanism

$g_{\mu\lambda} = i\cdot\sqrt{\frac{2\pi\omega_\lambda}{\hbar}}\;\vec{\Theta}_\mu\cdot\vec{u}_\lambda(\vec{x}_\mu)$: coupling constant which characterises the strength of the dipole interaction between the λ.th resonator mode and the μ.th laser-active atom

- The linear irreversible terms characterize the decay of the field amplitude $b_\lambda(t)$, the atomic polarizations $\alpha_\mu(t)$ and the atomic inversions $d_\mu(t)$. They are generated by the Maxwell equations and by ensemble considerations in non-relativistic quantum mechanics, respectively.

- From the nonlinear reversible terms one can read off the diverse feedback mechanisms:
  - according to the Maxwell equations the sum of the atomic polarizations drives the field amplitudes.
  - Because of the dipole interaction the atomic polarizations (atomic inversions) together with the field amplitudes drive the atomic inversions (atomic polarizations).

4.4. The Homogeneous Single-Mode Laser:

- In order to realistically describe the laser in the framework of the semiclassical laser theory one has to take into account approximatively 10^3 resonator modes and 10^{18} laser-active atoms as well as their respective nonlinear interactions. Because of the emerging complexity of the semiclassical laser equations it is not possible to obtain analytic solutions.

- For a ~~more~~ qualitative understanding of the laser it is sufficient to extract from the semiclassical laser equations a simple but physically reasonable model.

- To this end we impose the following physical approximations:
  - We regard the case that due to the mode selection by the mirrors (compare subsection 2.3) only one single mode becomes dominant in the resonator. For such a single-mode laser it is justified to omit the index λ of the various physical quantities.
  - We regard the case that all laser-active atoms are treated equally. For such a homogeneous laser it is justified to omit the index μ of the parameters which describe physical properties of the laser-active atoms.
  - We assume the resonance condition that the frequency ω of the dominant resonator mode coincides with the atomic frequency $\overline{\omega}$:

$$\omega = \overline{\omega}$$

With these physical approximations the semiclassical laser equations reduce to:

$$\dot b(t) = (-i\omega - \varkappa)\, b(t) - i g^* \sum_\mu \alpha_\mu(t)$$

$$\dot\alpha_\mu(t) = (-i\omega - \gamma_\perp)\,\alpha_\mu(t) + i g\; b(t)\, d_\mu(t)$$

$$\dot d_\mu(t) = \gamma_\parallel \{ d_0 - d_\mu(t) \} + 2i \{ g^* \alpha_\mu(t)\, b^*(t) - g\, \alpha_\mu^*(t)\, b(t) \}$$

- Then we introduce macroscopic variables by summing up the respective microscopic quantities:

electrical field: $E(t) = b(t)$

polarization: $P(t) = \sum_\mu \alpha_\mu(t)$

saturated inversion: $D(t) = \sum_\mu d_\mu(t)$

unsaturated inversion: $D_0 = N \cdot d_0$

Thereby N denotes the number of laser-active atoms. These definitions lead to macroscopic laser equations:

$$\dot E(t) = (-i\omega - \varkappa)\, E(t) - i g^*\, P(t)$$

$$\dot P(t) = (-i\omega - \gamma_\perp)\, P(t) + i g\; E(t)\, D(t)$$

$$\dot D(t) = \gamma_\parallel \{ D_0 - D(t) \} + 2i \{ g^* P(t)\, E^*(t) - g\, P^*(t)\, E(t) \}$$

- without loss of generality we now assume that the coupling constant g is real:

$$g^* = g$$

By performing the solution ansatz

$E(t) = \tilde E(t) \cdot e^{-i\omega t}$, $P(t) = \tilde P(t)\, i\, e^{-i\omega t}$, $D(t) = \tilde D(t)$

the complex-valued macroscopic laser equations for the functions $E(t), P(t), D(t)$ reduce to real-valued ones for the functions $\tilde E(t), \tilde P(t), \tilde D(t)$:

$$\dot{\tilde E}(t) e^{-i\omega t} - i\omega \tilde E(t) e^{-i\omega t} = (-i\omega - \varkappa) \tilde E(t) e^{-i\omega t} + g \tilde P(t) e^{-i\omega t}$$

$$\dot{\tilde P}(t) e^{-i\omega t} + \omega \tilde P(t) e^{-i\omega t} = (\omega - i\gamma_\perp) \tilde P(t) e^{-i\omega t} + i g \tilde E(t) e^{-i\omega t} \tilde D(t)$$

$$\dot{\tilde D}(t) = \gamma_\parallel \{ D_0 - \tilde D(t) \} + 2ig \{ i \tilde P(t) e^{-i\omega t} \tilde E(t) e^{+i\omega t} - (-i) \tilde P(t) e^{+i\omega t} \tilde E(t) e^{-i\omega t}$$

- When we omit the tilde we end up with the real-valued model equations for a homogeneous single-mode laser (Hermann Haken, 1975):

$$\frac{d}{dt} E(t) = -\varkappa E(t) + g\, P(t) \qquad (1)$$

$$\frac{d}{dt} P(t) = -\gamma_\perp P(t) + g\, E(t)\, D(t) \qquad (2)$$

$$\frac{d}{dt} D(t) = \gamma_\parallel \{ D_0 - D(t) \} - 4 g\, E(t)\, P(t) \qquad (3)$$

- By passing we note that these ordinary differential equations (1)–(3) of a homogeneous single-mode laser slightly differ from those mentioned in subsection 12.2. of "Hermann Haken: Synergetics – An Introduction". This discrepancy is due to an irrelevant rescaling of the involved physical quantities.

4.5. Relation to the Lorenz Equations:

- At a first glance the model equations (1)–(3) for a homogeneous single-mode laser seem to contain five parameters, namely $\varkappa, \gamma_\parallel, \gamma_\perp, g, D_0$. In order to determine how many of these parameters can be chosen independently we perform a rescaling of the variables by introducing some arbitrary constants T, A, B, C:

$$t = T \cdot t', \quad E(t) = \frac{1}{A} \cdot x\left(\frac{t}{T}\right), P(t) = \frac{1}{B}\, y\left(\frac{t}{T}\right), \quad D(t) = -\frac{1}{C}\, z\left(\frac{t}{T}\right) + D_0 \qquad (4)$$

Inserting the ansatz (4) in the model equations (1)–(3) we obtain:

$$\frac{d}{dt'} x(t') = -\varkappa T\, x(t') + \frac{gAT}{B}\, y(t') \qquad (5)$$

$$\frac{d}{dt'} y(t') = \frac{gBTD_0}{A}\, x(t') - \gamma_\perp T\, y(t') - \frac{gBT}{AC}\, z(t')\, x(t') \qquad (6)$$

$$\frac{d}{dt'} z(t') = -\gamma_\parallel T\, z(t') + \frac{4gCT}{AB}\, x(t')\, y(t') \qquad (7)$$

- In 1963 the meteorologist E.N. Lorenz analysed the Bénard instability in order to construct a simple model for the weather forecast. By truncating the respective Fourier expansions for the velocity and the temperature he derived the famous Lorenz equations:

$$\frac{d}{dt'} x(t') = -\sigma x(t') + \sigma y(t') \qquad (8)$$

$$\frac{d}{dt'} y(t') = r\, x(t') - y(t') - x(t')\, z(t') \qquad (9)$$

$$\frac{d}{dt} z(t') = -b\, z(t') + x(t')\, y(t') \qquad (10)$$

Thereby x denotes one Fourier coefficient of the velocity, whereas y and z stand for two different Fourier coefficients of the temperature. The parameter σ is the Prandtl number which is proportional to the reciprocal of the heat conductance of the regarded liquid. The parameter r represents the Rayleigh number which is proportional to the temperature gradient. Therefore the parameter r is regarded as the control parameter of the Bénard problem. The parameter b is an abbreviation for some constant.

- In the following we prove that the model equations (1)–(3) of the single-mode laser can be transformed by a suitable rescaling (4) to the Lorenz equations (8)–(10). To this end we have to show that the rescaled model equations (5)–(7) of the homogeneous single-mode laser are identical with the Lorenz equations (8)–(10) if we choose some proper rescaling constants T, A, B, C.
- Step 1:

$$T\gamma_\perp = 1 \quad \Longrightarrow \quad T = \frac{1}{\gamma_\perp} \qquad (11)$$

$$T\varkappa = \sigma \quad \xRightarrow{(11)} \quad \sigma = \frac{\varkappa}{\gamma_\perp} \qquad (12)$$

$$T\gamma_\parallel = b \quad \xRightarrow{(11)} \quad b = \frac{\gamma_\parallel}{\gamma_\perp} \qquad (13)$$

- Step 2:

$$\frac{gAT}{B} = \sigma \quad \xRightarrow{(11),(12)} \quad \frac{A}{B} = \frac{\varkappa}{g} \qquad (14)$$

$$\frac{gBTD_0}{A} = r \quad \xRightarrow{(11),(14)} \quad r = \frac{D_0}{D_0^I} \quad (15), \quad D_0^I = \frac{\gamma_\perp \varkappa}{g^2} \qquad (16)$$

$$\frac{gBT}{AC} = 1 \quad \xRightarrow{(11),(14),(16)} \quad C = \frac{1}{D_0^I} \qquad (17)$$

- Step 3:

$$\frac{4gCT}{AB} = 1 \quad \xRightarrow{(11),(16),(17)} \quad AB = \frac{4g^3}{\gamma_\perp^2 \varkappa} \qquad (18)$$

(14), (16), (18): $\quad B = \frac{2}{D_0^I} \quad (19), \quad A = \frac{2g}{\gamma_\perp} \quad (20)$

- We can summarise our result as follows:
  - By a rescaling procedure we can ~~extract~~ from the five dependent parameters $\varkappa, \gamma_\perp, \gamma_\parallel, g, D_0$ three independent, effective parameters σ, b, r defined in (12), (13), (15), (16).
  - Except of a rescaling procedure the model equations (1)–(3) of a homogeneous single-mode laser are identical with the Lorenz equations (8)–(10). Therefore we call the model (1)–(3) of a homogeneous single-mode laser the Haken-Lorenz model (compare the book "C.O. Weiss, R. Vilaseca: Dynamics of Lasers").

4.6. Relations to the Rate Equations:

- In the Haken-Lorenz model the typical time-scales of the electrical field $E(t)$, the polarization $P(t)$ and the inversion $D(t)$ are approximatively determined by the linear irreversible terms according to

$$\tau_E = \frac{1}{\varkappa}, \quad \tau_P = \frac{1}{\gamma_\perp}, \quad \tau_D = \frac{1}{\gamma_\parallel}$$

- In case of a homogeneous single-mode laser the typical time scales are given by

$$\tau_E = 10^{-6}\,s, \quad \tau_P = 10^{-12}\,s, \quad \tau_D = 10^{-10}\,s$$

- In a first approximation it is therefore justified to assume the following time-scale hierarchy:

$$\tau_P \gg \tau_E, \tau_D$$

The polarization $P(t)$ is regarded as a fast decreasing quantity whereas the electrical field $E(t)$ and the inversion $D(t)$ are the slowly varying quantities.

- These circumstances allow an adiabatic elimination of the polarization $P(t)$:

$$\dot{P}(t) = -\gamma_\perp P(t) + g\,E(t)D(t) \approx 0 \quad \Longrightarrow \quad P(t) \approx \frac{g}{\gamma_\perp}\cdot E(t)D(t)$$

- The corresponding evolution equations for the electrical field $E(t)$ and the inversion $D(t)$ then reduce to:

$$\dot{E}(t) = -\varkappa E(t) + \frac{g^2}{\gamma_\perp}\,E(t)D(t)$$

$$\dot{D}(t) = \gamma_\parallel\{D_0 - D(t)\} - \frac{4g^2}{\gamma_\perp}\,E(t)^2 D(t)$$

- As the photon number $n(t)$ measures the intensity of the electrical field $E(t)$ it is defined as the square of the electrical field

$$n(t) = E(t)^2$$

This redefinition changes the evolution equations:

$$\dot{n}(t) = -2\varkappa\, n(t) + \frac{2g^2}{\gamma_\perp}\,n(t)D(t)$$

$$\dot{D}(t) = \gamma_\parallel\{D_0 - D(t)\} - 2\cdot\frac{2g^2}{\gamma_\perp}\,n(t)D(t)$$

- An adiabatic elimination of the polarization in the Haken-Lorenz-Model for a homogeneous single-mode laser leads to the rate equations of a phenomenological laser theory (compare subsection 3.1.). Furthermore the identification of the respective parameters has the result

$$\gamma = \gamma_\parallel, \quad W = \frac{2g^2}{\gamma_\perp}$$

This result reveals how the phenomenological Einstein-coefficient W is related to microscopic quantities.

5. The Synergetic Method:

5.1. Evolution Equations:

- The synergetic method represents a universal approach for quantitatively analysing self-organisation processes which occur on macroscopic scales in various disciplines as, for instance, in physics, chemistry, biology, economy and sociology.
- The task of each individual discipline consists in proposing reasonable model equations which approximatively describe the evolution of the respective subsystems.
- If the state variables of the subsystems at time t can be integrated in a finite-dimensional state-vector $\vec{q}(t)$ then ordinary differential equations provide a typical example for such evolution equations:

$$\frac{d}{dt}\vec{q}(t) = \vec{N}(\vec{q}(t)) \qquad (1)$$

- Without emphasizing it explicitly in our notation we have to remember that the nonlinear vector field $\vec{N}(\vec{q})$ depends somehow on a set $\{\sigma\}$ of control-parameters. The consequences of these dependences will be discussed in subsection 5.4
- The evolution equations (1) can be generalised as follows:
  - In order to describe spatial inhomogeneities of the system one can use partial differential equations.
  - Stochastic forces can be included in the modelling by regarding stochastic differential equations.
  - If the finite propagation time of signals becomes important one can turn to delay or even functional differential equations.

5.2. Linear Stability Analysis:

- In practice a complete analysis of the evolution equations (1) is not possible. Therefore it becomes necessary to consider certain approximation methods in order to extract some relevant information.
- The linear stability analysis provides such a method by restricting the considerations to the vicinity of some reference state. In the simplest case the reference state $\vec{q}_{stat}$ is time-independent and is determined by

$$\vec{N}(\vec{q}_{stat}) = \vec{0} \qquad (2)$$

- We then introduce small deviations $\tilde{\vec{q}}(t)$ from this reference state $\vec{q}_{stat}$:

$$\vec{q}(t) = \vec{q}_{stat} + \tilde{\vec{q}}(t) \qquad (3)$$

- Inserting the ansatz (3) in the evolution equations (1) the Taylor expansion with respect to the small deviations $\tilde{\vec{q}}(t)$ reads up to the first order:

$$\frac{d}{dt}\tilde{\vec{q}}(t) = \underbrace{\vec{N}(\vec{q}_{stat})}_{\underset{(2)}{=}\ \vec{0}} + L\,\tilde{\vec{q}}(t) + \underbrace{\mathcal{O}(|\tilde{\vec{q}}(t)|^2)}_{\text{neglection}} \qquad (4)$$

definition of the matrix L: $$L_{ij} = \left.\frac{\partial N_i(\vec{q})}{\partial q_j}\right|_{\vec{q}=\vec{q}_{stat}} \qquad (5)$$

- By performing the solution ansatz

$$\tilde{\vec{q}}(t) = \vec{\phi}^{\lambda}\cdot e^{\lambda t} \qquad (6)$$

the linearized evolution equations (4) reduce to the right-hand eigenvalue problem of the matrix L where λ denotes the eigenvalue and $\vec{\phi}^{\lambda}$ the right-hand eigenvector:

$$L\,\vec{\phi}^{\lambda} = \lambda\,\vec{\phi}^{\lambda} \qquad (7)$$

- If the matrix L is not hermitian it is not sufficient to study only the right-hand eigenvalue problem (7) of the matrix L. In addition one has also to consider the left-hand eigenvalue problem

$$\vec{\psi}^{+\lambda}\,L = \lambda\,\vec{\psi}^{+\lambda} \qquad (8)$$

where $\vec{\psi}^{+\lambda}$ stands for the left-hand eigenvector (compare subsection 2.5. in "Hermann Haken: Advanced Synergetics"). The dagger + is in this context an abbreviation for both the transposition of the vector and the transition to the complex conjugated components.

- Non-trivial solutions of the eigenvalue problems (7) and (8) exist only if the eigenvalues λ solve the polynomial characteristic equation:

$$\det(L - \lambda I) = 0 \qquad (9)$$

If the state vector $\vec{q}(t)$ consists of n components then the characteristic equation (9) possesses n solutions $\lambda_1, \ldots, \lambda_n$.

- The left-hand eigenvectors $\vec{\psi}^{+\lambda_i}$ and the right-hand eigenvectors $\vec{\phi}^{\lambda_j}$ constitute a complete biorthonormal system:

biorthonormality relation: $$\vec{\psi}^{+\lambda_i}\cdot\vec{\phi}^{\lambda_j} = \delta_{ij} \qquad (10)$$

completeness relation: $$\sum_{i=1}^{n}\vec{\psi}^{+\lambda_i}\oplus\vec{\phi}^{\lambda_i} = I \qquad (11)$$

$\cdot$: inner product, $\oplus$: outer product

δ_{ij}: Kronecker symbol, I: $n\times n$-identity matrix

5.3. Mode Amplitudes:

- Now we turn again to the original nonlinear problem (1). Because of the completeness relation (11) the deviation of the state-vector $\vec{q}(t)$ from the reference state $\vec{q}_{stat}$ can be expanded with respect to the right-hand eigenvectors $\vec{\phi}^{\lambda_i}$:

$$\vec{q}(t) = \vec{q}_{stat} + \sum_{i=1}^{n} \xi_i(t) \vec{\phi}^{\lambda_i} \qquad (12)$$

In the following we call the right-hand eigenvectors $\vec{\phi}^{\lambda_i}$ modes and their expansion coefficients $\xi_i(t)$ mode amplitudes.

- The insertion of the expansion (12) in the nonlinear evolution equations (1) and a subsequent Taylor expansion of the nonlinear vector field $\vec{N}(\vec{q})$ around the reference state $\vec{q}_{stat}$ yields with (2) and (7):

$$\sum_{i=1}^{n} \dot{\xi}_i(t) \vec{\phi}^{\lambda_i} = \sum_{i=1}^{n} \lambda_i \xi_i(t) \vec{\phi}^{\lambda_i} + \qquad (13)$$

$$+ \sum_{r=2}^{\infty} \frac{1}{r!} \sum_{k_1=1}^{n} \cdots \sum_{k_r=1}^{n} \left.\frac{\partial^r \vec{N}(\vec{q})}{\partial q_{k_1} \cdots \partial q_{k_r}}\right|_{\vec{q}=\vec{q}_{stat}} \left\{\sum_{j_1=1}^{n} \xi_{j_1}(t) \phi_{k_1}^{\lambda_{j_1}}\right\} \cdots \left\{\sum_{j_r=1}^{n} \xi_{j_r}(t) \phi_{k_r}^{\lambda_{j_r}}\right\}$$

- The scalar product of the left-hand eigenvector $\vec{\psi}^{+\lambda_l}$ with equation (13) and the application of the biorthonormality relation (10) then leads to corresponding evolution equations for the mode amplitudes $\xi_i(t)$:

$$\dot{\xi}_i(t) = \lambda_i \xi_i(t) + \sum_{r=2}^{\infty} \sum_{j_1=1}^{n} \cdots \sum_{j_r=1}^{n} N^{(r)}_{i j_1 \cdots j_r} \xi_{j_1}(t) \cdots \xi_{j_r}(t) \qquad (14)$$

By passing we note that the original nonlinear evolution equation (1) for the state-vector $\vec{q}(t)$ and the evolution equations (14) for the mode-amplitudes $\xi_i(t)$ are equivalent as the coordinate transformation (12) between both is exact. But the advantage of the evolution equations (14) is that their linear part has been simplified by the aid of a diagonalisation procedure.

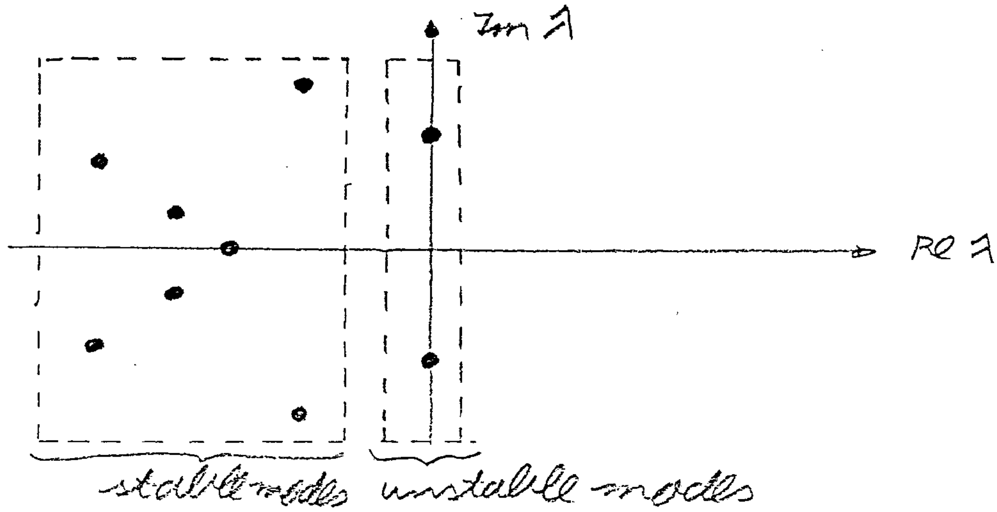
- The nonlinear coefficients $N^{(r)}_{i j_1 \cdots j_r}$ in the equations (14) can be calculated according to

$$N^{(r)}_{i j_1 \cdots j_r} = \frac{1}{r!} \sum_{l=1}^{n} \sum_{k_1=1}^{n} \cdots \sum_{k_r=1}^{n} \psi_l^{+\lambda_i} \left.\frac{\partial^r N_l(\vec{q})}{\partial q_{k_1} \cdots \partial q_{k_r}}\right|_{\vec{q}=\vec{q}_{stat}} \phi_{k_1}^{\lambda_{j_1}} \cdots \phi_{k_r}^{\lambda_{j_r}} \qquad (15)$$

In the case that the regarded system possesses some symmetries, group theoretical considerations make the calculation of the nonlinear coefficients $N^{(r)}_{i j_1 \cdots j_r}$ much easier.

5.4. Mode Decomposition:

- So far we have restricted our considerations to some fixed values of the control parameters $\{\sigma\}$. Now we discuss the case that the values of the control parameters $\{\sigma\}$ are changed.
- The equations (5) and (9) reveal that with the nonlinear vector field $\vec{N}(\vec{q})$ also the eigenvalues $\lambda_1, \dots, \lambda_n$ of the linearized problem depend somehow on the set $\{\sigma\}$ of control parameters.
- The reference state q_{stat} becomes unstable with respect to small perturbations if the real parts of some eigenvalues become positive. Therefore an instability occurs if the real parts of some eigenvalues vanish at critical values $\{\sigma_c\}$ of the control parameters:



- Such an instability gives rise to a mode decomposition where the unstable (stable) mode amplitudes are collected in the unstable (stable) vector $\vec{u}(t)$ ($\vec{s}(t)$):

$$\{\xi_i(t)\} \nearrow \vec{u}(t) = \begin{pmatrix} \xi_1(t) \\ \vdots \\ \xi_m(t) \end{pmatrix} \quad \text{with } |Re(\lambda_i)| \approx 0 \text{ for } i = 1, \dots, m$$

$$\{\xi_i(t)\} \searrow \vec{s}(t) = \begin{pmatrix} \xi_{m+1}(t) \\ \vdots \\ \xi_n(t) \end{pmatrix} \quad \text{with } |Re(\lambda_i)| \gg 0 \text{ for } i = m+1, \dots, n \tag{16}$$

- The respective eigenvalues of the unstable and stable modes can be collected in diagonal matrices:

$$\Lambda_u = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{pmatrix}, \quad \Lambda_s = \begin{pmatrix} \lambda_{m+1} & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \tag{17}$$

- With these redefinitions the evolution equations (14) for the mode amplitudes $\xi_i(t)$ can be rewritten in the following way

$$\begin{aligned} \dot{\vec{u}}(t) &= \Lambda_u \vec{u}(t) + \vec{N}_u(\vec{u}(t), \vec{s}(t)) \\ \dot{\vec{s}}(t) &= \Lambda_s \vec{s}(t) + \vec{N}_s(\vec{u}(t), \vec{s}(t)) \end{aligned} \tag{18}$$

5.5. Slaving Principle:

- According to equation (16) an instability is characterised by the inequality that the real parts of the unstable eigenvalues are much smaller than the real parts of the stable eigenvalues

$$|Re(\lambda_u)| \ll |Re(\lambda_s)| \qquad (18)$$

- By introducing the reciprocal of the real parts of the eigenvalues as some characteristic time-scales

$$\tau_u = \frac{1}{|Re(\lambda_u)|}, \quad \tau_s = \frac{1}{|Re(\lambda_s)|} \qquad (19)$$

the inequality (18) leads to a time scale hierarchy:

$$\tau_u \gg \tau_s \qquad (20)$$

- In comparison with the unstable modes the dynamics of the stable modes evolves very fast. This has the consequence that the stable modes quasiinstantaneously relax to an equilibrium value which is prescribed by the unstable modes

$$\vec{s}(t) = \vec{h}(\vec{u}(t)) \qquad (21)$$

This means that the stable modes $\vec{s}(t)$ are slaved by the unstable modes $\vec{u}(t)$ so that they can be no longer considered as independent degrees of freedom. Near the instability the dynamics of the system is completely prescribed by the unstable modes $\vec{u}(t)$. Therefore we call the unstable modes $\vec{u}(t)$ order parameters.

- Inserting the slaving ansatz (21) in the evolution equations (18) we obtain a relation which defines the center manifold $\vec{h} = \vec{h}(\vec{u})$ according to

$$\frac{d\vec{h}(\vec{u})}{d\vec{u}}\left\{\Lambda_u \vec{u} + \vec{N}_u(\vec{u}, \vec{h}(\vec{u}))\right\} = \Lambda_s \vec{h}(\vec{u}) + \vec{N}_s(\vec{u}, \vec{h}(\vec{u})) \qquad (2$$

- If the center manifold $\vec{h} = \vec{h}(\vec{u})$ is determined by solving equation (22) then the n-dimensional dynamics (1) finally reduces to the m-dimensional dynamics of the order parameter equations:

$$\dot{\vec{u}}(t) = \Lambda_u \vec{u}(t) + \vec{N}_u(\vec{u}(t), \vec{h}(\vec{u}(t))) \qquad (2$$

These order parameter equations completely describe the dynamics of the whole system near the regarded instability, that is for values of the control parameters $\{\sigma\}$ in the vicinity of the critical ones $\{\sigma_c\}$.

5.6. Determination of the Center Manifold:

- Now we discuss how it becomes possible to determine the center manifold $\vec{h} = \vec{h}(\vec{u})$ in the lowest order. This procedure justifies to some extent the heuristic adiabatic elimination described in subsection 3.4.
- To this end we expand the expressions $\vec{N}_u(\vec{u}, \vec{h}(\vec{u}))$ and $\vec{N}_s(\vec{u}, \vec{h}(\vec{u}))$ which appear in the equation (22) for the center manifold $\vec{h} = \vec{h}(\vec{u})$ into powers of the order parameter $\vec{u}$ and restrict ourselves to the lowest order:

$$N_{u,i}(\vec{u}, \vec{h}(\vec{u})) = \sum_{j_1=1}^{n} \cdots \sum_{j_r=1}^{n} N^{(r)}_{u,ij_1\ldots j_r} u_{j_1} \ldots u_{j_r} \tag{24}$$

$$N_{s,i}(\vec{u}, \vec{h}(\vec{u})) = \sum_{j_1=1}^{n} \cdots \sum_{j_r=1}^{n} N^{(r)}_{s,ij_1\ldots j_r} u_{j_1} \ldots u_{j_r}$$

- Then we can perform the ansatz that also the center manifold $\vec{h} = \vec{h}(\vec{u})$ possesses the lowest order r with respect to the order parameter $\vec{u}$:

$$h_i(\vec{u}) = \sum_{j_1=1}^{n} \cdots \sum_{j_r=1}^{n} H_{ij_1\ldots j_r} u_{j_1} \ldots u_{j_r} \tag{25}$$

- We insert the equations (24) and (25) into the determining equation (22) for the center manifold $\vec{h} = \vec{h}(\vec{u})$. Taking into account only the terms which are of the lowest order r we finally obtain for the expansion coefficients $H_{ij_1\ldots j_r}$ of the center manifold $\vec{h} = \vec{h}(\vec{u})$ the following result:

$$H_{ij_1\ldots j_r} = (-1) \left[\Lambda_s - (\lambda_{j_1} + \ldots + \lambda_{j_r}) I\right]^{-1}_{il} N^{(r)}_{s,lj_1\ldots j_r} \tag{26}$$

6. Application of the Synergetic Method to the Laser:

6.1. Stationary Solutions:

- This section is devoted to the application of the synergetic method (compare section 5) to the Haken – Lorenz model of a homogeneous single-mode laser (compare subsection 4.4):

$$\dot{E}(t) = N_E(E(t), P(t), D(t)) = -\varkappa E(t) + g P(t) \quad (1)$$

$$\dot{P}(t) = N_P(E(t), P(t), D(t)) = -\gamma_\perp P(t) + g E(t) D(t) \quad (2)$$

$$\dot{D}(t) = N_D(E(t), P(t), D(t)) = \gamma_\parallel \{D_0 - D(t)\} - 4g E(t) P(t) \quad (3)$$

- According to the general procedure we start with determining the stationary solutions of the evolution equations (1)-(3)

$$0 = -\varkappa E^0 + g P^0 \quad \Rightarrow \quad P^0 = \frac{\varkappa}{g} E^0 \quad (4)$$

$$0 = -\gamma_\perp P^0 + g E^0 D^0 \quad (5)$$

$$0 = \gamma_\parallel \{D_0 - D^0\} - 4g E^0 P^0 \quad (6)$$

- The insertion of (4) in (5) leads to: $E^0 \{ D^0 - \frac{\gamma_\perp \varkappa}{g^2} \} = 0$ (7)
From this we can read off that we have to consider two different cases.
- 1. case: $E^0_1 = 0 \xRightarrow{(4)} P^0_1 = 0 \xRightarrow{(6)} D^0_1 = D_0$ (8)
The first stationary solution exists for all values of the control parameter D_0. As the corresponding electrical field E^0_1 vanishes and as the spontaneous emission of radiation is ~~reflected in the framework of the semiclassical laser theory~~ (compare subsection 4.1) the first stationary solution ~~correspon~~ the incoherent lamp light.
- 2. case:

$$D^0_{2,3} = \frac{\gamma_\perp \varkappa}{g^2} \quad (9)$$

(4) and (9) in (6):

$$E^0_{2,3} = \pm \sqrt{\frac{\gamma_\parallel}{4\varkappa}\left(D_0 - \frac{\gamma_\perp \varkappa}{g^2}\right)} \quad (10)$$

(10) in (4):

$$P^0_{2,3} = \pm \sqrt{\frac{\gamma_\parallel \varkappa}{4g^2}\left(D_0 - \frac{\gamma_\perp \varkappa}{g^2}\right)} \quad . (11)$$

As the electrical field $E^0_{2,3}$ and the polarisation $P^0_{2,3}$ have to be real quantities the second and the third stationary solution only exist if the control parameter D_0 is larger than the critical value D_0^I which marks the first laser instability:

$$D_0 > D_0^I = \frac{\gamma_\perp \varkappa}{g^2} \quad (12)$$

The corresponding electrical field $E^0_{2,3}$ does not vanish so that the second and the third stationary solution can be interpreted as the coherent laser light.

- By passing we ~~remark~~ that the evolution equations (1)-(3) of the Haken – Lorenz model are invariant under the symmetry transformations

$$E(t) \rightarrow -E(t), \quad P(t) \rightarrow -P(t), \quad D(t) \rightarrow +D(t) \quad (13)$$

and that the stationary solutions (8)-(11) reflect this symmetry.

- In the next two subsections we perform a linear stability analysis around the first stationary solution (8). Among other things we thereby regard the special situation when the control parameter D_0 is in the neighbourhood of the first laser instability point D_0^I. To this end we introduce the dimensionless control parameter ε:

$$\varepsilon = \frac{D_0 - D_0^I}{D_0^I} \quad \overset{(12)}{\Longrightarrow} \quad D_0 = \frac{\gamma_\perp \varkappa}{g^2} + \frac{\gamma_\perp \varkappa}{g^2} \cdot \varepsilon \tag{14}$$

6.2. Eigenvalues:

- The Jacobi matrix of the nonlinear vector field of the Haken-Lorenz model (1)-(3) reads:

$$\mathcal{L}(E,P,D) = \begin{pmatrix} \frac{\partial N_E(E,P,D)}{\partial E} & \frac{\partial N_E(E,P,D)}{\partial P} & \frac{\partial N_E(E,P,D)}{\partial D} \\ \frac{\partial N_P(E,P,D)}{\partial E} & \frac{\partial N_P(E,P,D)}{\partial P} & \frac{\partial N_P(E,P,D)}{\partial D} \\ \frac{\partial N_D(E,P,D)}{\partial E} & \frac{\partial N_D(E,P,D)}{\partial P} & \frac{\partial N_D(E,P,D)}{\partial D} \end{pmatrix} = \begin{pmatrix} -\varkappa & +g & 0 \\ +gD & -\gamma_\perp & +gE \\ -4gP & -4gE & -\gamma_\parallel \end{pmatrix} \tag{15}$$

- The specialization to the first stationary solution results in

$$\mathcal{L}_1 = \mathcal{L}(E_1^0, P_1^0, D_1^0) = \begin{pmatrix} -\varkappa & +g & 0 \\ +gD_0 & -\gamma_\perp & 0 \\ 0 & 0 & -\gamma_\parallel \end{pmatrix} \tag{16}$$

- From this Jacobi matrix we obtain the following characteristic equation

$$\det(\mathcal{L}_1 - \lambda I) = -(\lambda + \gamma_\parallel)\{\lambda^2 - (\gamma_\perp + \varkappa)\lambda + \gamma_\perp \varkappa - g^2 D_0\} = 0 \tag{17}$$

whose solutions represent the eigenvalues:

$$\lambda_{1,2} = -\frac{\gamma_\perp + \varkappa}{2} \pm \sqrt{\left(\frac{\gamma_\perp + \varkappa}{2}\right)^2 - \gamma_\perp \varkappa + g^2 D_0}\,, \quad \lambda_3 = -\gamma_\parallel \tag{18}$$

- Inserting (14) in (18) we can determine the dependence of the respective eigenvalues on the dimensionless control parameter ε. A Taylor expansion with respect to the dimensionless control parameter ε leads in lowest order to:

$$\lambda_1 = -\frac{\gamma_\perp + \varkappa}{2} + \frac{\gamma_\perp + \varkappa}{2} \cdot \sqrt{1 + \frac{4\gamma_\perp \varkappa}{(\gamma_\perp + \varkappa)^2} \cdot \varepsilon} = \frac{\gamma_\perp \varkappa}{\gamma_\perp + \varkappa} \cdot \varepsilon + \mathcal{O}(\varepsilon^2)$$

$$\lambda_2 = -\frac{\gamma_\perp + \varkappa}{2} - \frac{\gamma_\perp + \varkappa}{2} \cdot \sqrt{1 + \frac{4\gamma_\perp \varkappa}{(\gamma_\perp + \varkappa)^2} \cdot \varepsilon} = -(\gamma_\perp + \varkappa) + \mathcal{O}(\varepsilon) \tag{19}$$

$$\lambda_3 = -\gamma_\parallel$$

- In the case $\varepsilon < 0$ all three eigenvalues are negative so that the first stationary solution (8) is stable. But in the case $\varepsilon > 0$ the first eigenvalue becomes positive and the first stationary solution unstable. From this we conclude that the first laser instability represents a Pitchfork bifurcation.

6.3. Eigenvectors:

- For the right-hand eigenvectors $\vec{\phi}^{\lambda_i}$ we obtain:

$$\mathcal{L}_1 \vec{\phi}^{\lambda_i} = \lambda_i \vec{\phi}^{\lambda_i} \; ; \; i = 1, 2, 3 \qquad (20)$$

$$i = 3: \quad \vec{\phi}^{\lambda_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad (21)$$

$$i = 1, 2: \; (-\kappa - \lambda_i)\phi_1^{\lambda_i} + g\, \phi_2^{\lambda_i} = 0 \Rightarrow \vec{\phi}^{\lambda_i} = N_i \begin{pmatrix} g \\ \lambda_i + \kappa \\ 0 \end{pmatrix}$$

- Equivalently the left-hand eigenvectors $\vec{\psi}^{+\lambda_i}$ result in

$$\vec{\psi}^{+\lambda_i} \mathcal{L}_1 = \lambda_i \vec{\psi}^{+\lambda_i} \; ; \; i = 1, 2, 3 \qquad (22)$$

$$i = 3: \quad \vec{\psi}^{+\lambda_3} = (0, 0, 1) \qquad (23)$$

$$i = 1, 2: \; (-\kappa - \lambda_i)\psi_1^{+\lambda_i} + g D_0 \psi_2^{+\lambda_i} = 0 \Rightarrow \vec{\psi}^{+\lambda_i} = N_i^+ (g D_0, \lambda_i + \kappa, 0)$$

- The respective normalization constants N_i and N_i^+ for $i = 1, 2$ are determined from the normalization condition:

$$\vec{\psi}^{+\lambda_i} \cdot \vec{\phi}^{\lambda_i} = N_i N_i^+ \left\{ g^2 D_0 + (\lambda_i + \kappa)^2 \right\} = 1 \qquad (24)$$

obviously the normalization constants N_i and N_i^+ are not unique. By choosing $N_i = N_i^+$ we obtain from (24):

$$N_i = \frac{1}{\sqrt{g^2 D_0 + (\lambda_i + \kappa)^2}} \qquad (25)$$

- For the dependence of the normalization constants N_i on the dimensionless control parameter ε we obtain:

$$N_1 \stackrel{(14),(25)}{=\!=} \frac{1}{\sqrt{\gamma_\perp \kappa (1+\varepsilon) + (\lambda_1 + \kappa)^2}} \stackrel{(19)}{=\!=} \frac{1}{\sqrt{\gamma_\perp \kappa + \kappa^2}} + \mathcal{O}(\varepsilon) \qquad (26)$$

$$N_2 \stackrel{(14),(25)}{=\!=} \frac{1}{\sqrt{\gamma_\perp \kappa (1+\varepsilon) + (\lambda_2 + \kappa)^2}} \stackrel{(19)}{=\!=} \frac{1}{\sqrt{\gamma_\perp \kappa + \gamma_\perp^2}} + \mathcal{O}(\varepsilon)$$

- We remark that the left-hand eigenvectors $\vec{\psi}^{+\lambda_i}$ are not identical with the transposition of the right-hand eigen-vectors $\vec{\phi}^{\lambda_i}$ as the matrix $\mathcal{L}_1$ is not symmetric.
- We note that the left-hand and right-hand eigenvectors indeed fulfill the biorthonormality relation:

$$\vec{\psi}^{+\lambda_i} \cdot \vec{\phi}^{\lambda_j} = \delta_{ij} \; ; \; i, j = 1, 2, 3 \qquad (27)$$

6.4. Mode Amplitudes:

- After the linear stability analysis we now return to the original nonlinear Laser-Lorenz model (1)-(3). We introduce the deviations from the first stationary solution:

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} E(t) - E_1^0 \\ P(t) - P_1^0 \\ D(t) - D_1^0 \end{pmatrix} \stackrel{(8)}{=} \begin{pmatrix} E(t) \\ P(t) \\ D(t) - D_0 \end{pmatrix} \qquad (28)$$

- According to (1)-(3) and (28) the evolution equations for the deviations then read as follows:

$$\begin{aligned} \dot{x}_1(t) &= -\varkappa x_1(t) + g x_2(t) \\ \dot{x}_2(t) &= g D_0 x_1(t) - \gamma_\perp x_2(t) \qquad + g x_1(t) x_3(t) \\ \dot{x}_3(t) &= \qquad - \gamma_\parallel x_3(t) - 4 g x_1(t) x_2(t) \end{aligned} \qquad (29)$$

- A compact notation for these evolution equations is given by

$$\dot{x}_i(t) = \sum_{j=1}^{3} L_{ij} x_j(t) + \sum_{j=1}^{3} \sum_{k=1}^{3} B_{ijk} x_j(t) x_k(t) \qquad (30)$$

L_{ij} denote the components of the matrix $\mathcal{L}_1$ defined in (16) and the nonlinear coefficients B_{ijk} are given by

$$B_{213} = B_{231} = \frac{g}{2}, \quad B_{312} = B_{321} = -2g, \quad B_{ijk} = B_{ikj} = 0 \text{ for other } ijk \quad ($$

- The deviations $x_i(t)$ from the first stationary solution are expanded with respect to the collective modes $\vec{\phi}^{\lambda_\beta}$ which have been determined in the linear stability analysis. Thereby we introduce the respective mode amplitudes $\xi_\beta(t)$:

$$\vec{x}(t) = \sum_{\beta=1}^{3} \xi_\beta(t) \vec{\phi}^{\lambda_\beta} \quad \Rightarrow \quad x_i(t) = \sum_{\beta=1}^{3} \xi_\beta(t) \phi_i^{\lambda_\beta} \qquad (3$$

- Inserting this ansatz in the evolution equations (30) for the deviations $x_i(t)$ we obtain

$$\sum_{\beta=1}^{3} \dot{\xi}_\beta(t) \phi_i^{\lambda_\beta} = \sum_{\beta=1}^{3} \underbrace{\left\{ \sum_{j=1}^{3} L_{ij} \phi_j^{\lambda_\beta} \right\}}_{\stackrel{(20)}{=} \lambda_\beta \cdot \phi_i^{\lambda_\beta}} \xi_\beta(t) + \sum_{\beta=1}^{3} \sum_{\gamma=1}^{3} \left\{ \sum_{j=1}^{3} \sum_{k=1}^{3} B_{ijk} \phi_j^{\lambda_\beta} \phi_k^{\lambda_\gamma} \right\} \xi_\beta(t) \xi_\gamma(t)$$

- We multiply this equation with $\psi_i^{+\lambda_\alpha}$ and perform the summation over the index i:

$$\sum_{\beta=1}^{3} \underbrace{\left\{ \sum_{i=1}^{3} \psi_i^{+\lambda_\alpha} \phi_i^{\lambda_\beta} \right\}}_{\stackrel{(27)}{=} \delta_{\alpha\beta}} \dot{\xi}_\beta(t) = \sum_{\beta=1}^{3} \lambda_\beta \underbrace{\left\{ \sum_{i=1}^{3} \psi_i^{+\lambda_\alpha} \phi_i^{\lambda_\beta} \right\}}_{\stackrel{(27)}{=} \delta_{\alpha\beta}} \xi_\beta(t)$$

$$+ \sum_{\beta=1}^{3} \sum_{\gamma=1}^{3} \left\{ \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \psi_i^{+\lambda_\alpha} B_{ijk} \phi_j^{\lambda_\beta} \phi_k^{\lambda_\gamma} \right\} \xi_\beta(t) \xi_\gamma(t)$$

- In this way we can deduce evolution equations for the mode amplitudes $\xi_\alpha(t)$:

$$\dot{\xi}_\alpha(t) = \lambda_\alpha \xi_\alpha(t) + \sum_{\beta=1}^{3}\sum_{\gamma=1}^{3} A_{\alpha\beta\gamma}\, \xi_\beta(t)\, \xi_\gamma(t) \qquad (33)$$

where the nonlinear coefficients $A_{\alpha\beta\gamma}$ are defined by

$$A_{\alpha\beta\gamma} = A_{\alpha\gamma\beta} = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \psi_i^{+\lambda_\alpha} B_{ijk}\, \phi_j^{\lambda_\beta} \phi_k^{\lambda_\gamma} \qquad (34)$$

Taking into account (31) the nonlinear coefficients $A_{\alpha\beta\gamma}$ reduce to

$$A_{\alpha\beta\gamma} = B_{213}\, \psi_2^{+\lambda_\alpha} \{ \phi_1^{\lambda_\beta} \phi_3^{\lambda_\gamma} + \phi_3^{\lambda_\beta} \phi_1^{\lambda_\gamma} \} + B_{312}\, \psi_3^{+\lambda_\alpha} \{ \phi_1^{\lambda_\beta} \phi_2^{\lambda_\gamma} + \phi_2^{\lambda_\beta} \phi_1^{\lambda_\gamma} \} \qquad (35)$$

By applying the equations (19), (21), (23), (26), (31), (33), (35) we now determine the evolution equations for $\xi_1(t)$, $\xi_2(t)$, $\xi_3(t)$ up to the lowest order in the dimensionless control parameter ε.

- 1. case: $\alpha = 1$

$$A_{1\beta\gamma} = A_{1\gamma\beta} = B_{213}\, \psi_2^{+\lambda_1} \{ \phi_1^{\lambda_\beta} \phi_3^{\lambda_\gamma} + \phi_3^{\lambda_\beta} \phi_1^{\lambda_\gamma} \}$$

$$A_{113} = A_{131} = B_{213}\, \psi_2^{+\lambda_1} \phi_1^{\lambda_1} \phi_3^{\lambda_3} = \frac{g}{2} N_1 (\lambda_1 + \varkappa) N_1 g = \frac{g^2}{2(\gamma_\perp + \varkappa)} + O(\varepsilon)$$

$$A_{123} = A_{132} = B_{213}\, \psi_2^{+\lambda_1} \phi_1^{\lambda_2} \phi_3^{\lambda_3} = \frac{g}{2} N_1 (\lambda_1 + \varkappa) N_2 g = \sqrt{\frac{\varkappa}{\gamma_\perp}} \cdot \frac{g^2}{2(\gamma_\perp + \varkappa)} + O(\varepsilon)$$

$$A_{111} = A_{112} = A_{121} = A_{122} = A_{133} = 0$$

$$\dot{\xi}_1(t) = \lambda_1(\varepsilon)\xi_1(t) + a_1(\varepsilon)\xi_1(t)\xi_3(t) + a_2(\varepsilon)\xi_2(t)\xi_3(t) \qquad (36)$$

$$\lambda_1(\varepsilon) = \frac{\gamma_\perp \varkappa}{\gamma_\perp + \varkappa}\varepsilon + O(\varepsilon^2), \quad a_1(\varepsilon) = \frac{g^2}{\gamma_\perp + \varkappa} + O(\varepsilon), \quad a_2(\varepsilon) = \sqrt{\frac{\varkappa}{\gamma_\perp}} \cdot \frac{g^2}{\gamma_\perp + \varkappa} + O(\varepsilon) \qquad (37)$$

- 2. case: $\alpha = 2$

$$A_{2\beta\gamma} = A_{2\gamma\beta} = B_{213}\, \psi_2^{+\lambda_2} \{ \phi_1^{\lambda_\beta} \phi_3^{\lambda_\gamma} + \phi_3^{\lambda_\beta} \phi_1^{\lambda_\gamma} \}$$

$$A_{213} = A_{231} = B_{213}\, \psi_2^{+\lambda_2} \phi_1^{\lambda_1} \phi_3^{\lambda_3} = \frac{g}{2} N_2 (\lambda_2 + \varkappa) N_1 g = -\sqrt{\frac{\gamma_\perp}{\varkappa}} \cdot \frac{g^2}{2(\gamma_\perp + \varkappa)} + O(\varepsilon)$$

$$A_{223} = A_{232} = B_{213}\, \psi_2^{+\lambda_2} \phi_1^{\lambda_2} \phi_3^{\lambda_3} = \frac{g}{2} N_2 (\lambda_2 + \varkappa) N_2 g = -\frac{g^2}{2(\gamma_\perp + \varkappa)} + O(\varepsilon)$$

$$A_{211} = A_{212} = A_{221} = A_{222} = A_{233} = 0$$

$$\dot{\xi}_2(t) = \lambda_2(\varepsilon)\xi_2(t) + b_1(\varepsilon)\xi_1(t)\xi_3(t) + b_2(\varepsilon)\xi_2(t)\xi_3(t) \qquad (38)$$

$$\lambda_2(\varepsilon) = -(\gamma_\perp + \varkappa) + O(\varepsilon), \quad b_1(\varepsilon) = -\sqrt{\frac{\gamma_\perp}{\varkappa}} \cdot \frac{g^2}{\gamma_\perp + \varkappa} + O(\varepsilon), \quad b_2(\varepsilon) = -\frac{g^2}{\gamma_\perp + \varkappa} + O(\varepsilon) \qquad (39)$$

- 3. case: $\alpha = 3$

$$A_{3\beta\gamma} = A_{3\gamma\beta} = B_{312}\, \psi_3^{+\lambda_3} \{ \phi_1^{\lambda_\beta} \phi_2^{\lambda_\gamma} + \phi_2^{\lambda_\beta} \phi_1^{\lambda_\gamma} \}$$

$$A_{311} = 2 B_{312}\, \phi_1^{\lambda_1} \phi_2^{\lambda_1} = 2(-2) g N_1 g N_1 (\lambda_1 + \varkappa) = -\frac{4g^2}{\gamma_\perp + \varkappa} + O(\varepsilon)$$

$$A_{312} = A_{321} = B_{312} \{ \phi_1^{\lambda_2} \phi_2^{\lambda_1} + \phi_1^{\lambda_1} \phi_2^{\lambda_2} \} = -\frac{1}{\sqrt{\gamma_\perp \varkappa}} \cdot \frac{2g^2(\varkappa - \gamma_\perp)}{\varkappa + \gamma_\perp} + O(\varepsilon)$$

$$A_{322} = 2 B_{312}\, \phi_1^{\lambda_2} \phi_2^{\lambda_2} = 2(-2g) N_2 g N_2 (\lambda_2 + \varkappa) = \frac{4g^2}{\varkappa + \gamma_\perp} + O(\varepsilon)$$

$$\dot{\xi}_3(t) = \lambda_3(\varepsilon)\xi_3(t) + c_1(\varepsilon)\xi_1(t)^2 + c_2(\varepsilon)\xi_1(t)\xi_2(t) + c_3(\varepsilon)\xi_2(t)^2 \qquad (40)$$

$$\lambda_3(\varepsilon) = -\gamma_\parallel, \quad c_1(\varepsilon) = \frac{-4g^2}{\gamma_\perp + \varkappa} + O(\varepsilon), \quad c_2(\varepsilon) = \frac{4g^2(\gamma_\perp - \varkappa)}{\sqrt{\gamma_\perp \varkappa}\,(\varkappa + \gamma_\perp)} + O(\varepsilon), \quad c_3(\varepsilon) = \frac{4g^2}{\varkappa + \gamma_\perp} + O(\varepsilon) \qquad (41)$$

6.5. Order Parameter Equation:

- In the following we perform a systematic self-consistency procedure in order to obtain the order parameter equation of the Haken-Lorenz model in the vicinity of the first laser instability. By doing so we rigorously prove for this example that the adiabatic elimination procedure (compare subsectio[n] 3.4) turns out to be a direct consequence of such a systematic self-consistency procedure.
- We start with assuming that the mode amplitudes $\xi_1(t)$, $\xi_2(t)$, $\xi_3(t)$ depend on the dimensionless control parameter ε according to

$$\xi_1(t) = \mathcal{O}(\varepsilon^{1/2}), \quad \xi_2(t) = \mathcal{O}(\varepsilon^{3/2}), \quad \xi_3(t) = \mathcal{O}(\varepsilon^{1}) \qquad (4$$

- Then we regard the evolution equations (36)-(41) for the mode amplitudes $\xi_1(t)$, $\xi_2(t)$, $\xi_3(t)$ and restrict ourselves to the terms with the lowest order in the dimensionless control parameter ε:

$$\dot{\xi}_1(t) = \underbrace{\frac{\gamma_\perp \varkappa}{\gamma_\perp + \varkappa}\,\varepsilon\,\xi_1(t)}_{=\mathcal{O}(\varepsilon^{3/2})} + \underbrace{\frac{g^2}{\gamma_\perp + \varkappa}\,\xi_1(t)\,\xi_3(t)}_{=\mathcal{O}(\varepsilon^{3/2})} + \underbrace{\cancel{\sqrt{\frac{\varkappa}{\gamma_\perp}}\,\frac{g^2}{\gamma_\perp + \varkappa}\,\xi_2(t)\,\xi_3(t)}}_{=\mathcal{O}(\varepsilon^{5/2})} \quad (4$$

$$\dot{\xi}_2(t) = \underbrace{-(\gamma_\perp + \varkappa)\,\xi_2(t)}_{=\mathcal{O}(\varepsilon^{3/2})} \underbrace{- \sqrt{\frac{\gamma_\perp}{\varkappa}}\,\frac{g^2}{\gamma_\perp + \varkappa}\,\xi_1(t)\,\xi_3(t)}_{=\mathcal{O}(\varepsilon^{3/2})} \underbrace{- \cancel{\frac{g^2}{\gamma_\perp + \varkappa}\,\xi_2(t)\,\xi_3(t)}}_{=\mathcal{O}(\varepsilon^{5/2})} \quad (4$$

$$\dot{\xi}_3(t) = \underbrace{-\gamma_\parallel\,\xi_3(t)}_{=\mathcal{O}(\varepsilon^{1})} \underbrace{- \frac{4g^2}{\gamma_\perp + \varkappa}\,\xi_1(t)^2}_{=\mathcal{O}(\varepsilon^{1})} + \underbrace{\frac{4g^2(\gamma_\perp - \varkappa)}{\sqrt{\gamma_\perp \varkappa}\,(\gamma_\perp + \varkappa)}\,\xi_1(t)\,\xi_2(t)}_{=\mathcal{O}(\varepsilon^{2})} + \underbrace{\cancel{\frac{4g^2}{\varkappa + \gamma_\perp}\,\xi_2(t)^2}}_{=\mathcal{O}(\varepsilon^{3})} \quad (4$$

- $\xi_2(t)$ and $\xi_3(t)$ represent the fast varying mode amplitudes. Due to the slaving principle of synergetics they are enslaved by the slowly varying mode amplitude $\xi_1(t)$:

$$\xi_2(t) = h_2(\xi_1(t)), \quad \xi_3(t) = h_3(\xi_1(t)) \qquad (4$$

- For the time derivatives of the mode amplitudes $\xi_2(t)$ and $\xi_3(t)$ we then obtain in the lowest order of the dimensionless control parameter ε:

$$\dot{\xi}_2(t) \overset{(46)}{=\!=} \frac{\partial h_2(\xi_1(t))}{\partial \xi_1(t)} \cdot \dot{\xi}_1(t) \overset{(42),(43)}{=\!=\!=\!=} \mathcal{O}(\varepsilon^{5/2}) \qquad (47)$$

$$\dot{\xi}_3(t) \overset{(46)}{=\!=} \frac{\partial h_3(\xi_1(t))}{\partial \xi_1(t)} \cdot \dot{\xi}_1(t) \overset{(42),(43)}{=\!=\!=\!=} \mathcal{O}(\varepsilon^{2}) \qquad (48)$$

- From (44) - (48) we read off that the time derivatives of the mode amplitudes $\xi_2(t)$ and $\xi_3(t)$ represent higher order terms in ε so that they can be neglected. Therefore we conclude from the self-consistency procedure that the mode amplitudes $\xi_2(t)$ and $\xi_3(t)$ can be eliminated adiabatically:

$$-(\gamma_\perp+\kappa)h_2(\xi_1)-\sqrt{\frac{\gamma_\perp}{\kappa}}\cdot\frac{g^2}{\gamma_\perp+\kappa}\,\xi_1\,h_3(\xi_1)=0 \qquad (49)$$

$$-\gamma_\parallel\,h_3(\xi_1)-\frac{4g^2}{\gamma_\perp+\kappa}\,\xi_1^2=0 \qquad (50)$$

From this the center manifolds $h_2=h_2(\xi_1)$ and $h_3=h_3(\xi_1)$ are determined according to

$$h_2(\xi_1)=\sqrt{\frac{\gamma_\perp}{\kappa}}\cdot\frac{4g^4}{\gamma_\parallel(\gamma_\perp+\kappa)^3}\,\xi_1^3 \qquad (51)$$

$$h_3(\xi_1)=-\frac{4g^2}{\gamma_\parallel(\gamma_\perp+\kappa)}\,\xi_1^2 \qquad (52)$$

- Inserting the center manifold $h_3=h_3(\xi_1)$ from (52) in the evolution equation (43) for the mode amplitude $\xi_1(t)$ one results in the order parameter equation of the Haken-Lorenz model in the vicinity of the first laser instability. It represents the normal form of a pitchfork bifurcation:

$$\dot{\xi}_1(t)=\frac{\gamma_\perp\kappa}{\gamma_\perp+\kappa}\,\varepsilon\,\xi_1(t)-\frac{4g^4}{\gamma_\parallel(\gamma_\perp+\kappa)^2}\,\xi_1(t)^3 \qquad (53)$$

- The stationary solutions of this order parameter equation are given by:

$$\xi_1^1=0 \qquad ;\ \text{for all } \varepsilon \qquad (54)$$

$$\xi_1^{2,3}=\pm\sqrt{\frac{\gamma_\parallel\gamma_\perp\kappa(\gamma_\perp+\kappa)}{4g^4}\,\varepsilon}=\mathcal{O}(\varepsilon^{1/2}) \qquad ;\ \text{for } \varepsilon\geq 0$$

With (46), (51), (52) and (54) the above-made assumptions (4...
of the self-consistency procedure turn out to be justified
- In order to check the order parameter equation (53) we re-construct the second and the third stationary solution of the Haken-Lorenz model in the lowest order of the dimensionless control parameter ε:

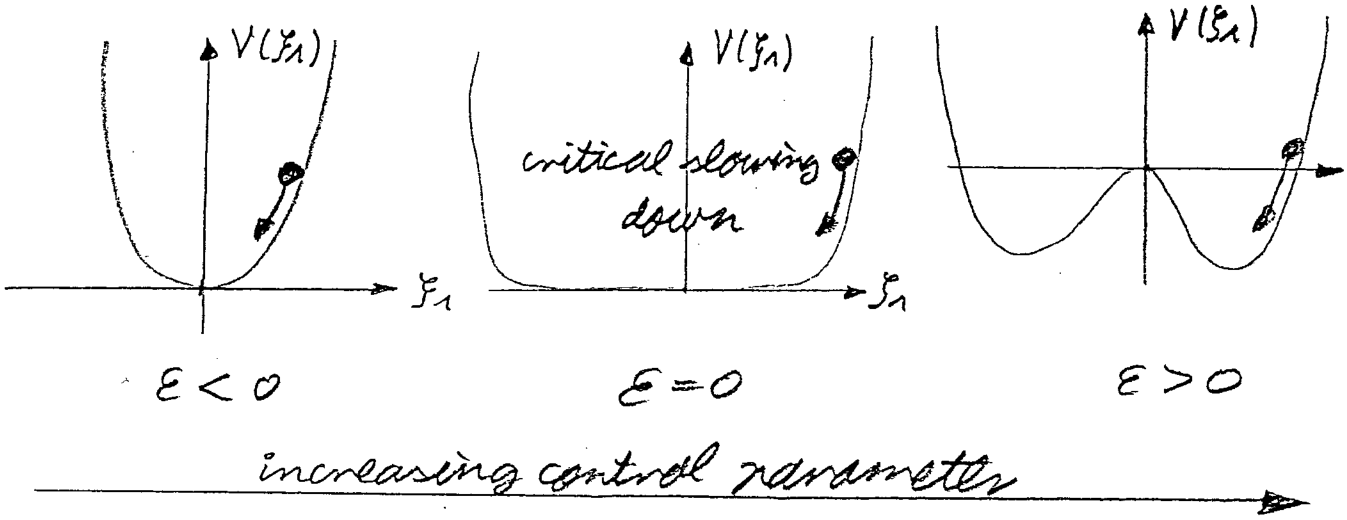
$$\begin{pmatrix} E^0_{2,3} \\ P^0_{2,3} \\ D^0_{2,3} \end{pmatrix} \underset{=}{\overset{(28),(32),(42)}{}} \begin{pmatrix} 0 \\ 0 \\ D_0 \end{pmatrix} + \xi_1^{2,3}\,\vec{\varphi}_{\lambda_1} \underset{=}{\overset{(21),(26),(54)}{}} \begin{pmatrix} \pm\frac{\sqrt{\gamma_\parallel\gamma_\perp}}{2g}\cdot\sqrt{\varepsilon} \\ \pm\frac{\sqrt{\gamma_\parallel\gamma_\perp}\,\kappa}{2g^2}\sqrt{\varepsilon} \\ D_0 \end{pmatrix} \qquad (5\ldots$$

This indeed coincides in the lowest order in ε with (9)-(11), (1...

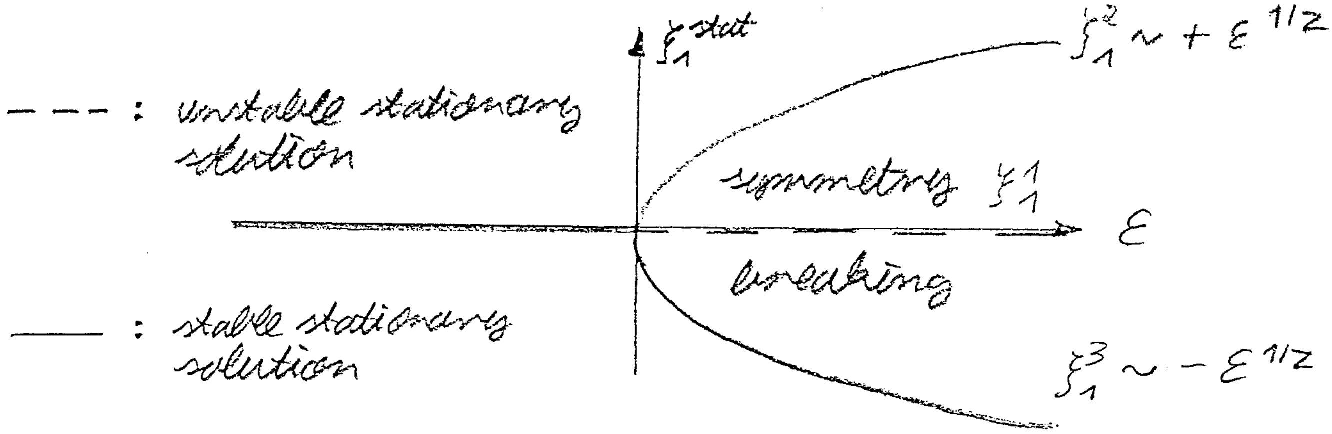
- The stability of the stationary solutions (54) of the order parameter equation (53) can be discussed by introducing a potential (compare subsection 3.4.):

$$\dot{\xi}_1(t) = -\frac{\partial V(\xi_1(t))}{\partial \xi_1(t)} \xrightarrow{(53)} V(\xi_1) = -\frac{\gamma_\perp \varkappa}{2(\gamma_\perp+\varkappa)}\,\xi_1^2 + \frac{g^4}{\gamma_\parallel(\gamma_\perp+\varkappa)^2}\,\xi_1^4 \qquad (56)$$

The shape of the potential then changes with increasing the dimensionless control parameter ε:



- From this we read off the corresponding bifurcation diagram of the Pitchfork bifurcation:



6.6. Second Laser Instability:

- Now we turn to a linear stability analysis around the second and third stationary solution (9)-(11). In this case the Jacobi matrix (15) reduces to:

$$\mathcal{L}_{2,3} = \mathcal{L}(E^0_{2,3}, P^0_{2,3}, D^0_{2,3}) = \begin{pmatrix} -\varkappa & g & 0 \\ \frac{\gamma_\perp \varkappa}{g} & -\gamma_\perp & g E^0_{2,3} \\ -4\varkappa E^0_{2,3} & -4 g E^0_{2,3} & -\gamma_\parallel \end{pmatrix} \qquad (57)$$

- For the corresponding characteristic equation we then obtain by applying (10):

$$\det(\mathcal{L}_{2,3} - \lambda I) = 0$$

$$P(\lambda)=\lambda^3+(\gamma_\parallel+\gamma_\perp+\varkappa)\lambda^2+\left\{\gamma_\parallel\gamma_\perp+\gamma_\parallel\varkappa+\frac{\gamma_\parallel g^2}{\varkappa}\left(D_0-\frac{\gamma_\perp\varkappa}{g^2}\right)\right\}\lambda+2g^2\gamma_\parallel\left(D_0-\frac{\gamma_\perp\varkappa}{g^2}\right)=0 \quad (58)$$

- We note that the characteristic equation is the same for the second and the third stationary solution (9)–(11). This has the consequence that both stationary solutions possess the same stability for a given value of the control parameter D_0.
- It remains for us to determine the three solutions $\lambda_1, \lambda_2, \lambda_3$ of this characteristic equation. Thereby we can use the following relation between the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and the cubic polynomial $P(\lambda)$:

$$P(\lambda)=(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)$$
$$=\lambda^3-(\lambda_1+\lambda_2+\lambda_3)\lambda^2+(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)\lambda-\lambda_1\lambda_2\lambda_3 \quad (59)$$

A comparison between (58) and (59) results in

$$\lambda_1\lambda_2\lambda_3=-2g^2\gamma_\parallel\left(D_0-\frac{\gamma_\perp\varkappa}{g^2}\right) \quad (60)$$

As we assume that the control parameter D_0 is larger than the first laser instability point $D_0^I=\frac{\gamma_\perp\varkappa}{g^2}$ we deduce from (60) the inequality

$$\lambda_1\lambda_2\lambda_3<0 \quad (61)$$

- We investigate now the conditions under which a second laser instability could occur.
- 1. case: $\lambda_1, \lambda_2, \lambda_3$ real

i.) $\lambda_1, \lambda_2, \lambda_3 < 0$: all modes remain stable

ii.) $\lambda_1, \lambda_2 > 0$, $\lambda_3 < 0$: two modes become simultaneously unstable

Instability point according to ii.): $\lambda_1=\lambda_2=0$, $\lambda_3<0$

$$\Rightarrow \quad P(\lambda)=\lambda^2(\lambda-\lambda_3)=\lambda^3-\lambda_3\cdot\lambda^2 \quad (62)$$

A comparison between (58) and (62) shows that there exists no value for the control parameter D_0 where this second laser instability occurs in this case.

- 2. case: $\lambda_1=\lambda_2^*$ complex, λ_3 real

Instability point: $\lambda_1=+i\omega$, $\lambda_2=\lambda_1^*=-i\omega$, $\lambda_3=\delta<0$

$$\Rightarrow P(\lambda)=(\lambda-i\omega)(\lambda+i\omega)(\lambda-\delta)=\lambda^3-\delta\lambda^2+\omega^2\lambda-\omega^2\delta \quad (63)$$

A comparison with the general cubic polynomial

$$P(\lambda)=\lambda^3+a\lambda^2+b\lambda+c \quad (64)$$

leads to the following relation between the parameters a, b, c

$$c=ab \quad (65)$$

Inserting the special parameter values a, b, c from (58) and (64) we obtain from (65) a determining relation for the second laser instability:

$$2g^2\left(D_0 - \frac{\gamma_\perp \varkappa}{g^2}\right) = (\gamma_\parallel + \gamma_\perp + \varkappa)\left\{\gamma_\parallel\gamma_\perp + \gamma_\parallel\varkappa + \frac{\gamma_\parallel g^2}{\varkappa}\left(D_0 - \frac{\gamma_\perp\varkappa}{g^2}\right)\right\}$$

$$\Longrightarrow \quad D_0^{II} = D_0^{I} \cdot \frac{\varkappa(\varkappa + \gamma_\parallel + 3\gamma_\perp)}{\gamma_\perp(\varkappa - \gamma_\parallel - \gamma_\perp)}, \quad D_0^{I} = \frac{\gamma_\perp \varkappa}{g^2} \qquad (66)$$

- From the condition $D_0^{II} > 0$ we deduce that the second laser instability only occurs in a bad cavity where the resonator losses are large:

$$\varkappa > \gamma_\parallel + \gamma_\perp \qquad (67)$$

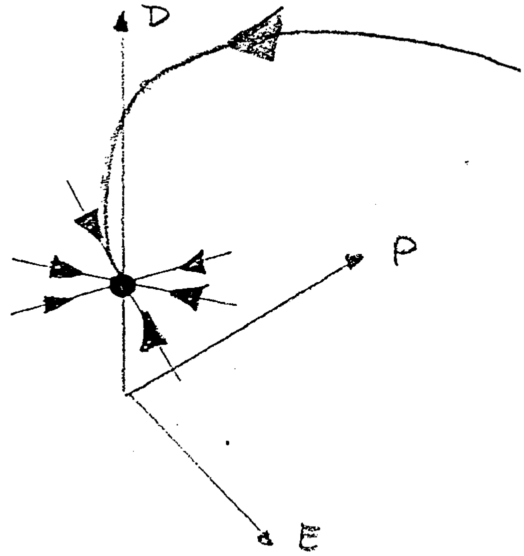
This is the reason why it is difficult to experimentally detect the second laser instability (compare the book "C.O. Weiss, R. Vilaseca: Dynamics of Lasers").

6.7. Summary:

- Corresponding to the Haken-Lorenz model the homogeneous single-mode laser reveals different macroscopic ordered states when the pumping D_0 is increased unspecifically (compare the next page).
- In the case $D_0 < D_0^{I}$, that is below the first laser instability, the electrical field vanishes in the long-time limit. As the Haken-Lorenz model neglects the spontaneous emission of radiation this final laser state has to be identified with the incoherent light output of an ordinary lamp.
- In the case $D_0^{I} < D_0 < D_0^{II}$, that is just above the first laser instability, the electrical field approaches a finite value which is prescribed by the pumping D_0 according to (10). This represents a regular laser activity where the light output occurs in a coherent way.
- In the case $D_0^{II} < D_0$, that is above the second laser instability, the electrical field shows a chaotic time evolution so that the light output is irregular and unpredictable (compare subsection 4.5 and the Lorenz attractor on the last page).

Overview:

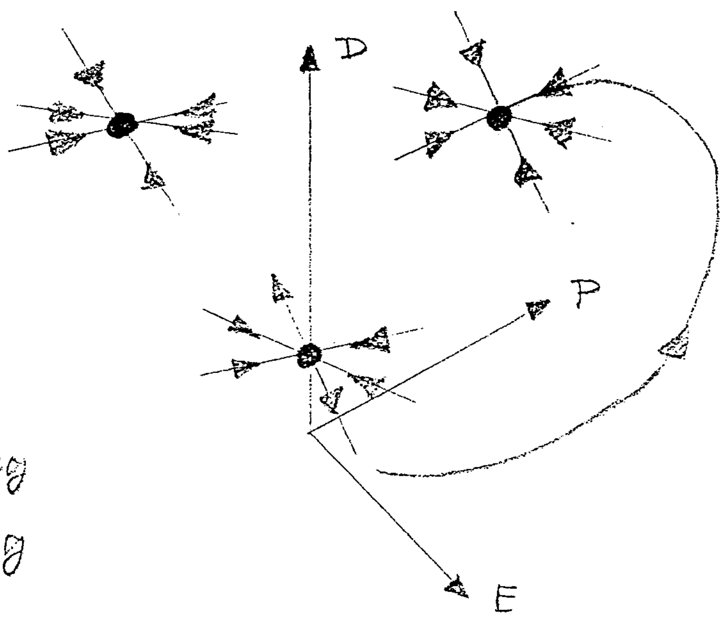
$D_0 < D_0^{I}$



stable node

increasing pumping

$D_0^{I} < D_0 < D_0^{II}$

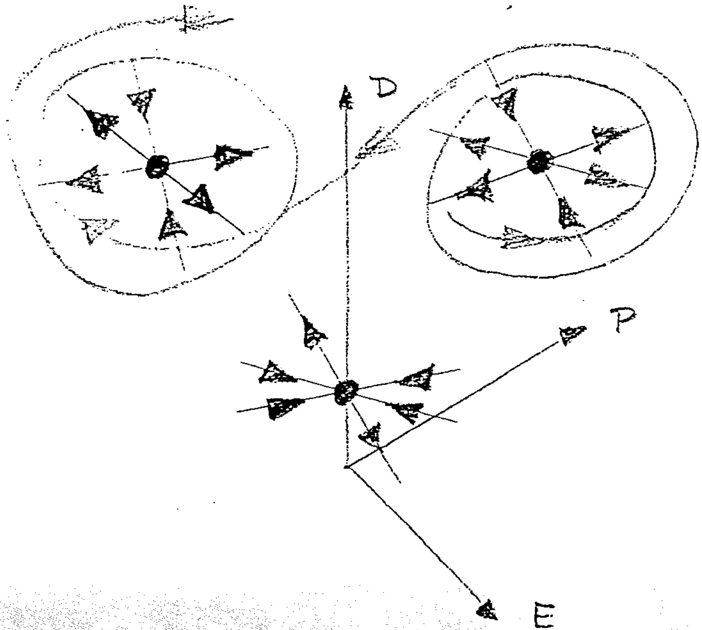


stable nodes

saddle point

increasing pumping

$D_0^{II} < D_0$



unstable foci

saddle point

lorenz attractor

$$\dot{x} = \sigma (y - x)$$
$$\dot{y} = x (r - z) - y$$
$$\dot{z} = x y - b z$$

$r = 45.92$

$\sigma = 16.00$

$b = 4.00$