NOVEL GEOMETRIC GAUGE INVARIANCE OF AUTOPARALLELS*

H. KLEINERT AND A. PELSTER

Institut für Theoretische Physik
Freie Universität Berlin, Arnimallee 14
D-14195 Berlin, Germany
e-mail kleinert@physik.fu-berlin.de,
http://www.physik.fu-berlin.de/~kleinert
e-mail pelster@physik.fu-berlin.de,
http://www.physik.fu-berlin.de/~pelster

(Received January 14, 1998)

Dedicated to Andrzej Trautman in honour of his 64^{th} birthday

We draw attention to a novel type of geometric gauge invariance relating the autoparallel equations of motion in different Riemann-Cartan spacetimes with each other. The novelty lies in the fact that the equations of motion are invariant even though the actions are not. As an application we use this gauge transformation to map the action of a spinless point particle in a Riemann-Cartan spacetime with a gradient torsion to a purely Riemann spacetime, in which the initial torsion appears as a nongeometric external field. By extremizing the transformed action in the usual way, we obtain the same autoparallel equations of motion as those derived in the initial spacetime with torsion via a recently-discovered variational principle.

PACS numbers: 02.40. –k, 04.20. Fy

1. Introduction

Einstein's theory of general relativity predicts correctly all post-Newtonian experiments in our solar system as well as some effects of strong gravitational fields observed in binary systems of neutron stars [1]. The theory has, however, two unsatisfactory properties. One is the somewhat academic fact

* Presented at the Workshop on Gauge Theories of Gravitation, Jadwisin, Poland, September 4–10, 1997.

that gravity cannot be quantized in a renormalizable way [2], but only as an effective theory which is unable to predict short-distance gravitational phenomena at a scale much shorter than the Planck length $l_P \approx 10^{-32}$cm [3]. Since this length scale is extremely small, the problem is not very serious for present-day physics. The other unsatisfactory property which has received much attention is of an esthetical nature. Since all elementary-particle forces known so far are mediated by local gauge fields, one would like to describe also the geometric theory of gravity as a local Poincaré gauge symmetry. This would naturally introduce torsion into the geometry [4–7], thus extending Riemann spacetime of the Einstein theory to Riemann-Cartan spacetime. Therein parallelograms exhibit both an angular and a closure failure due to curvature and torsion, respectively. So far, such spacetimes have only found applications in the theory of plastic flow and material fatigue [7–12], where curvature and torsion are produced by disclinations and dislocations.

The last two decades have seen a detailed elaboration of the gauge-theoretic formulations of gravity in spacetimes with torsion, most notably the Einstein-Cartan theory [6,13]. The latter has the appealing feature that Einstein's equation stating the proportionality between curvature and energy momentum tensor of matter is extended by a corresponding one involving torsion and spin density. In this theory spinless particles do not create torsion, and since usually only particles which are sources of a field can also be influenced by this field, it has generally been believed that trajectories of spinless particles are not affected by torsion in spacetime, *i.e.*, that spinless point particles move along *geodesics*, the shortest curves in the spacetime. With this bias the relativity community happily embraced Hehl's derivation [14] of such trajectories from hitherto accepted gravitational field equations as has been recently reviewed in [15].

This belief and its field theoretic derivation have recently been challenged by one of us [16–23]. It was pointed out that the invariance under general coordinate transformations of general relativity used by Einstein to find the laws of nature in a curved geometry may be replaced by a more efficient *nonholonomic mapping principle*, which is moreover predicitve for spacetimes with torsion. This new principle was originally discovered for the purpose of transforming nonrelativistic path integrals correctly from flat space to spaces with torsion [18], where it played the role of a *quantum equivalence principle*. Evidence for its correctness was derived from its essential role in solving the path integral of the hydrogen atom, where the nonholonomic Kustaanheimo-Stiefel transformation was used [18].

Applying the nonholonomic mapping principle to the variational procedure has the important consequence that the Euler-Lagrange equations of spinless point particles receive an additional force which depends on the torsion tensor $S_{\mu\nu}{}^{\lambda}(q)$ [21]. If τ denotes an arbitrary parameter of the trajectory $q^{\lambda}(\tau)$, the modified Euler-Lagrange equations read

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{q}^{\lambda}(\tau)} - \frac{\partial L}{\partial q^{\lambda}(\tau)} = -2S_{\lambda\mu}{}^{\nu}\left(q(\tau)\right)\dot{q}^{\mu}(\tau)\frac{\partial L}{\partial \dot{q}^{\nu}(\tau)}\,. \tag{1}$$

The modification on the right-hand side has its origin in the closure failure of parallelograms in spacetimes with torsion, which can be accounted for by a noncommutativity of nonholonomic variations δ with the parameter derivative $d_{\tau} = d/d\tau$ of the trajectory $q^{\lambda}(\tau)$ [21]:

$$\delta d_{\tau} q^{\lambda}(\tau) - d_{\tau}\,\delta q^{\lambda}(\tau) = 2S_{\mu\nu}{}^{\lambda}(q(\tau))\dot{q}^{\mu}(\tau)\,\delta q^{\nu}(\tau)\,. \tag{2}$$

For the free-particle Lagrangian [24]

$$L(q^{\lambda}, \dot{q}^{\lambda}) = -Mc\sqrt{g_{\lambda\mu}(q)\dot{q}^{\lambda}\dot{q}^{\mu}} \tag{3}$$

with the proper time

$$ds(\tau) = \sqrt{g_{\lambda\mu}(q(\tau))dq^{\lambda}(\tau)dq^{\mu}(\tau)}\,, \tag{4}$$

the modified Euler-Lagrange equation (1) becomes explicitly

$$\begin{aligned}\ddot{q}^{\lambda}(s) + g^{\lambda\kappa}(q(s))\left[\partial_{\mu}g_{\nu\kappa}(q(s)) - \frac{1}{2}\partial_{\kappa}g_{\mu\nu}(q(s))\right]\dot{q}^{\mu}(s)\dot{q}^{\nu}(s) \\ = -2S_{\mu\nu}{}^{\lambda}(q(s))\dot{q}^{\mu}(s)\dot{q}^{\nu}(s)\,.\end{aligned} \tag{5}$$

Here we can use the decomposition of the affine connection $\Gamma_{\mu\nu}{}^{\lambda}(q)$ in a Riemann-Cartan spacetime [25]

$$\Gamma_{\mu\nu}{}^{\lambda}(q) = \overline{\Gamma}_{\mu\nu}{}^{\lambda}(q) + K_{\mu\nu}{}^{\lambda}(q)\,, \tag{6}$$

where the first term is the Christoffel connection

$$\overline{\Gamma}_{\mu\nu}{}^{\lambda}(q) = \frac{1}{2}g^{\lambda\kappa}(q)\left[\partial_{\mu}g_{\nu\kappa}(q) + \partial_{\nu}g_{\kappa\mu}(q) - \partial_{\kappa}g_{\mu\nu}(q)\right] \tag{7}$$

depending only on the metric $g_{\mu\nu}(q)$, while the second one is the contortion tensor

$$K_{\mu\nu}{}^{\lambda}(q) = S_{\mu\nu}{}^{\lambda}(q) - S_{\nu}{}^{\lambda}{}_{\mu}(q) + S^{\lambda}{}_{\mu\nu}(q) \tag{8}$$

representing a combination of the torsion tensor $S_{\mu\nu}{}^{\lambda}(q)$. Then (5) reduces to the straightest lines or *autoparallels* in a Riemann-Cartan spacetime:

$$\ddot{q}^{\lambda}(s) + \Gamma_{\mu\nu}{}^{\lambda}(q(s))\dot{q}^{\mu}(s)\dot{q}^{\nu}(s) = 0\,. \tag{9}$$

The coupling of spinless point particles to torsion in (9) has a single geometrical interpretation. To this end we observe that the natural covariant derivative of an arbitrary vector field $V^{\lambda}(q)$

$$D_{\mu}V^{\lambda}(q) = \partial_{\mu}V^{\lambda}(q) + \Gamma_{\mu\nu}{}^{\lambda}(q)V^{\nu}(q) \tag{10}$$

contains the full affine connection (6) rather then the Christoffel connection (7). Thus we may define a covariant derivative of $V^{\lambda}(q(s))$ with respect to the proper time s as

$$\frac{D}{Ds}V^{\lambda}(q(s)) = \frac{d}{ds}V^{\lambda}(q(s)) + \Gamma_{\mu\nu}{}^{\lambda}(q(s))\dot{q}^{\mu}(s)V^{\nu}(q(s))\,, \tag{11}$$

so that the autoparallel equation (9) reads simply [26]

$$\frac{D}{Ds}\dot{q}^{\lambda}(s) = 0\,. \tag{12}$$

Therefore the transition from the Minkowski to the Riemann-Cartan spacetime corresponds to the substitution of the total derivative with respect to the proper time d/ds by the covariant one D/Ds. This means that the free spinless point particle is minimally coupled to the gravitational field in a Riemann-Cartan spacetime.

2. Novel type of geometric gauge invariance

We now turn to the essential point of our lecture that the new variational procedure [21] implies a novel kind of geometric gauge invariance of the autoparallel trajectories although the action is *not* invariant. It turns out that a change in the action can be compensated by a corresponding change in the closure failure at the endpoints of the paths. This gauge invariance is based on transformations which were introduced in 1982 by two research groups in a different context [27, 28]. Following their notation, we transform metric and torsion simultaneously, the first conformally, the second by adding a gradient term. The Riemann–Cartan curvature tensor remains invariant under this transformation which merely shifts part of the geometry from the Riemann to the torsion part. We shall demonstrate that although the particle action changes under this transformation, initial and final actions yield

the same autoparallel particle trajectory, thus making the two geometries indistinguishable from each other by any measuring process involving spinless point-like test particles. In particular the gauge invariance will allow us to relate the action of point particles in a specific family of Riemann-Cartan spacetimes in which torsion arises from the gradient of a scalar field to that in a Riemann spacetime. In the transformed action, the scalar field plays the role of a nongeometric external field, and particle trajectories can be derived via the traditional action principle, yielding the same autoparallels as in the initial spacetime with gradient torsion via the new action principle.

2.1. Geometry transformations

In a Riemann-Cartan geometry [25, 26], the metric $g_{\mu\nu}(q)$ and the affine connection $\Gamma_{\mu\nu}{}^{\lambda}(q)$ are not independent of each other. The decomposition (6)–(8) implies that they must satisfy the metricity condition

$$D_\lambda g_{\mu\nu}(q) = \partial_\lambda g_{\mu\nu}(q) - \Gamma_{\lambda\mu}{}^{\kappa}(q) g_{\kappa\nu}(q) - \Gamma_{\lambda\nu}{}^{\kappa}(q) g_{\mu\kappa}(q) = 0\,. \tag{13}$$

The metric $g_{\mu\nu}(q)$ and the torsion tensor $S_{\mu\nu}{}^{\lambda}(q)$, however, represent independent quantities characterizing the Riemann-Cartan geometry.

The fundamental gauge invariance of a Riemann-Cartan geometry contains two ingredients [27, 28]. Following Weyl [29], who postulated that no physical phenomenon should depend on the choice of dimensional units, we transform the metric conformally to

$$\tilde{g}_{\mu\nu}(q) = e^{2\sigma(q)} g_{\mu\nu}(q) \tag{14}$$

with an arbitrary scalar function $\sigma(q)$. We supplement this transformation by a change of the torsion tensor according to

$$\tilde{S}_{\mu\nu}{}^{\lambda}(q) = S_{\mu\nu}{}^{\lambda}(q) + \frac{1}{2}\left[\delta_\nu{}^{\lambda}\partial_\mu\sigma(q) - \delta_\mu{}^{\lambda}\partial_\nu\sigma(q)\right]. \tag{15}$$

For the Christoffel connection (7) and the contortion tensor (8), these transformations imply

$$\tilde{\overline{\Gamma}}_{\mu\nu}{}^{\lambda}(q) = \overline{\Gamma}_{\mu\nu}{}^{\lambda}(q) + \delta_\nu{}^{\lambda}\partial_\mu\sigma(q) + \delta_\mu{}^{\lambda}\partial_\nu\sigma(q) - g_{\mu\nu}(q) g^{\lambda\kappa}(q)\partial_\kappa\sigma(q), \tag{16}$$

$$\tilde{K}_{\mu\nu}{}^{\lambda}(q) = K_{\mu\nu}{}^{\lambda}(q) - \delta_\mu{}^{\lambda}\partial_\nu\sigma(q) + g_{\mu\nu}(q) g^{\lambda\kappa}(q)\partial_\kappa\sigma(q), \tag{17}$$

respectively. In the affine connection (6), the transformations (16), (17) almost compensate each other, resulting only in an additional gradient term:

$$\tilde{\Gamma}_{\mu\nu}{}^{\lambda}(q) = \Gamma_{\mu\nu}{}^{\lambda}(q) + \delta_\nu{}^{\lambda}\partial_\mu\sigma(q)\,. \tag{18}$$

In the Riemann-Cartan geometry, this leaves the curvature tensor

$$R_{\mu\nu\kappa}{}^{\lambda}(q) = \partial_\mu \Gamma_{\nu\kappa}{}^{\lambda}(q) - \partial_\nu \Gamma_{\mu\kappa}{}^{\lambda}(q) + \Gamma_{\nu\kappa}{}^{\rho}(q)\Gamma_{\mu\rho}{}^{\lambda}(q) - \Gamma_{\mu\kappa}{}^{\rho}(q)\Gamma_{\nu\rho}{}^{\lambda}(q) \quad (19)$$

invariant:

$$\tilde{R}_{\mu\nu\kappa}{}^{\lambda}(q) = R_{\mu\nu\kappa}{}^{\lambda}(q). \quad (20)$$

The associated curvature scalar,

$$R(q) = g^{\nu\kappa}(q) R_{\mu\nu\kappa}{}^{\mu}(q), \quad (21)$$

however, is changed by a conformal transformation of the inverse metric following from (14), so one obtains

$$\tilde{R}(q) = e^{-2\sigma(q)} R(q)\,. \quad (22)$$

Due to (18), the geometry transformations (14), (15) change covariant derivatives of the metric by a conformal factor

$$\tilde{D}_\lambda \tilde{g}_{\mu\nu}(q) = e^{2\sigma(q)}\, D_\lambda g_{\mu\nu}(q)\,. \quad (23)$$

As a consequence, covariant derivatives with respect to general coordinate transformations also remain covariant under the geometry tranformations. For this reason, the metricity condition (13) holds also after a geometry transformation

$$\tilde{D}_\lambda \tilde{g}_{\mu\nu}(q) = 0\,, \quad (24)$$

guaranteeing the gauge invariance of the Riemann-Cartan geometry.

2.2. Application to autoparallel particle trajectories

We now prove our principal result that the pair of geometry transformations (14) and (15) between different Riemann-Cartan spacetimes leaves trajectories of spinless point particles invariant. To this end we start with the Lagrangian

$$\tilde{L}(q^\lambda, \dot{q}^\lambda) = -Mc\sqrt{\tilde{g}_{\lambda\mu}(q)\dot{q}^\lambda \dot{q}^\mu} \quad (25)$$

and with the torsion tensor $\tilde{S}_{\mu\nu}{}^{\lambda}(q)$, where the new action principle leads to a modified Euler-Lagrange equation like (1):

$$\frac{d}{d\tau}\frac{\partial \tilde{L}}{\partial \dot{q}^\lambda(\tau)} - \frac{\partial \tilde{L}}{\partial q^\lambda(\tau)} = -2\tilde{S}_{\lambda\mu}{}^{\nu}\left(q(\tau)\right)\dot{q}^\mu(\tau)\frac{\partial \tilde{L}}{\partial \dot{q}^\nu(\tau)}\,. \quad (26)$$

Inserting (25) in (26) and defining the proper time $\tilde{s}$ in analogy to (4)

$$d\tilde{s}(\tau) = \sqrt{\tilde{g}_{\lambda\mu}(q(\tau))dq^{\lambda}(\tau)dq^{\mu}(\tau)}\,, \tag{27}$$

we obtain the autoparallel equation associated with the affine connection $\tilde{\Gamma}_{\mu\nu}{}^{\lambda}(q)$

$$\ddot{q}^{\lambda}(\tilde{s}) + \tilde{\Gamma}_{\mu\nu}{}^{\lambda}(q(\tilde{s}))\dot{q}^{\mu}(\tilde{s})\dot{q}^{\nu}(\tilde{s}) = 0\,. \tag{28}$$

After reexpressing this equation in terms of the original affine connection $\Gamma_{\mu\nu}{}^{\lambda}(q)$ with (18) and defining the proper time s via the relation

$$\frac{ds}{d\tilde{s}} = e^{-\sigma(q)} \tag{29}$$

due to (4), (14) and (27), we finally get the autoparallel equation (9). Let us remark that nonintegrable time transformations of the type (29) have been extensively used in solving various classical, quantum mechanical, and stochastic problems [18, 30–35].

Alternatively we could have also derived this result by rewriting the Lagrangian (25) according to (14):

$$\tilde{L}(q^{\lambda}, \dot{q}^{\lambda}) = -Mc\,e^{\sigma(q)}\sqrt{g_{\lambda\mu}(q)\dot{q}^{\lambda}\dot{q}^{\mu}}. \tag{30}$$

Inserting (15) and (30) in (26), all terms involving the scalar function $\sigma(q)$ cancel each other, thus leading again to the autoparallel equation (9).

An important insight is gained by considering the special case of a Riemann-Cartan spacetime where the entire torsion tensor arises from the gradient of a scalar field $\sigma(q)$:

$$S_{\mu\nu}{}^{\lambda}(q) = \tfrac{1}{2}\left[\delta_{\mu}{}^{\lambda}\partial_{\nu}\sigma(q) - \delta_{\nu}{}^{\lambda}\partial_{\mu}\sigma(q)\right]. \tag{31}$$

From the new action principle then follows the autoparallel equation

$$\ddot{q}^{\lambda}(s) + \overline{\Gamma}_{\mu\nu}{}^{\lambda}(q(s))\dot{q}^{\mu}(s)\dot{q}^{\nu}(s) = -\dot{\sigma}(q(s))\dot{q}^{\lambda}(s) + g^{\lambda\kappa}(q(s))\partial_{\kappa}\sigma(q(s))\,. \tag{32}$$

However, after performing the geometric gauge transformation to a purely Riemannian spacetime, the torsion scalar $\sigma(q)$ appears as a nongeometric external field in the modified Lagrangian (30), so that we find precisely the same equation of motion via the usual action principle.

3. Conclusion and outlook

We showed that the local geometry transformations (14), (15) with the property (18) induce mappings between the autoparallel trajectories in different Riemann-Cartan spacetimes. This represents an unusual type of symmetry since the equations of motion remain invariant whereas the respective actions change. The different terms in the actions are compensated by corresponding contributions from the endpoints of the paths which take into account the closure failure in the presence of torsion. We consider this novel gauge invariance as an interesting property of autoparallel trajectories which should help giving us hints on how to construct the proper field equations of a theory of gravitation with torsion.

We are grateful to Drs. G. Barnich, H. von Borzeskowski, A. Schakel, S.V. Shabanov and to the graduate student C. Maulbetsch for many stimulating discussions.

REFERENCES

[1] R.A. Hulse, J.H. Taylor, *Astrophys. J.* **195**, L51 (1975).

[2] M.H. Goroff, A. Sagnotti, *Nucl. Phys.* **B266**, 709 (1986).

[3] E. D'Hoker, S. Weinberg, *Phys. Rev.* **D50**, 6050 (1994).

[4] R. Utiyama, *Phys. Rev.* **101**, 1597 (1956).

[5] T.W.B. Kibble, *J. Math. Phys.* **2**, 212 (1961).

[6] F.W. Hehl, P. von der Heyde, G.D. Kerlick, J.M. Nester, *Rev. Mod. Phys.* **48**, 393 (1976).

[7] H. Kleinert, *Gauge Fields in Condensed Matter, Vol. II, Stresses and Defects*, World Scientific, Singapore 1989.

[8] B.A. Bilby, R. Bullough, E. Smith, *Proc. R. Soc. London* **A231**, 263 (1955).

[9] E. Kröner, *Continuum Theory of Defects*, in R. Balian *et al.*, *Physics of Defects, Les Houches, Session XXXV, 1980*, North-Holland, 215 (1981).

[10] E. Kröner, *Int. J. Theor. Phys.* **29**, 1219 (1990).

[11] R. Bausch, R. Schmitz, Ł.A. Turski, *Phys. Rev. Lett.* **73**, 2382 (1994).

[12] R. Bausch, R. Schmitz, Ł.A. Turski, *Z. Phys.* **B97**, 171 (1995).

[13] F.W. Hehl, J.D. McCrea, E.W. Mielke, Y. Ne'eman, *Phys. Rep.* **258**, 1 (1995).

[14] F.W. Hehl, *Phys. Lett.* **A36**, 225 (1971).

[15] V. de Sabbata, Evidence for Torsion in Gravity?, Lecture at the XV International School of Cosmology and Gravitation in Erice/Italy (1997).

[16] H. Kleinert, *Mod. Phys. Lett.* **A4**, 2329 (1989).

[17] H. Kleinert, Quantum Equivalence Principle for Path Integrals in Spaces with Curvature and Torsion, Lecture at the XXV International Symposium Ahrenshoop in Gosen/Germany (1991).

[18] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics and Polymer Physics*, World Scientific, Singapore, Second Edition, 1995.

[19] H. Kleinert, Quantum Equivalence Principle, Lecture presented at the Summer School Functional Integration: Basics and Applications in Cargèse/France (1996).

[20] P. Fiziev, H. Kleinert, *Europhys. Lett.* **35**, 241 (1996) (hep-th/9503074).

[21] H. Kleinert, A. Pelster, Berlin preprint (1996) (gr-qc/9605028).

[22] H. Kleinert, S.V. Shabanov, Berlin preprint (1997) (gr-qc/9709067).

[23] H. Kleinert, Nonholonomic Mapping Principle for Classical and Quantum Mechanics in Spaces with Curvature and Torsion, FU-Berlin preprint (1997) (APS E-Print aps1997sep03_002).

[24] S. Weinberg, *Gravitation and Cosmology – Principles and Applications of the General Theory of Relativity*, John Wiley & Sons, New York 1972.

[25] J.A. Schouten, *Ricci-Calculus*, Springer, Berlin, Second Edition, 1954.

[26] E. Schrödinger, *Space-Time Structure*, Cambridge University Press, 1954.

[27] Yu.N. Obukhov, *Phys. Lett.* **A 90**, 13 (1982).

[28] T. Dereli, R.W. Tucker, *J. Phys.* **A15**, L7 (1982).

[29] H. Weyl, *Raum-Zeit-Materie*, Springer, Berlin 1920.

[30] H. Duru, H. Kleinert, *Phys. Lett.* **B84**, 185 (1979).

[31] H. Duru, H. Kleinert, *Fortschr. Phys.* **30**, 401 (1982).

[32] A. Pelster, A. Wunderlin, *Z. Phys.* **B89**, 373 (1992).

[33] A. Pelster, *Zur Theorie und Anwendung nichtintegrabler Raum-Zeit-Transformationen in der klassischen Mechanik und in der Quantenmechanik*, Dissertation, Universität Stuttgart, Shaker-Verlag, 1996.

[34] A. Pelster, H. Kleinert, Transformation Properties of Classical and Quantum Laws Under Some Nonholonomic Transformations; Lecture at the V International Conference on *Path Integrals from meV to MeV* in Dubna/Russia (1996), Proceedings ed. by V.S. Yarurin, M.A. Smondyrev, publishing department of Joint Institute for Nuclear Research, Dubna 1996, p. 187, (quant-ph/9608037).

[35] A. Pelster, H. Kleinert, *Phys. Rev. Lett.* **78**, 565 (1997) (cond-mat/9608120).