

Boundary-layer theory, strong-coupling series, and large-order behavior

Carl M. Bender^{a)}

Department of Physics, Washington University, St. Louis, Missouri 63130

Axel Pelster^{b)} and Florian Weissbach^{c)}

*Institut für Theoretische Physik, Freie Universität Berlin,
Arnimallee 14, 14195 Berlin, Germany*

(Received 15 March 2002; accepted for publication 26 April 2002)

The introduction of a lattice converts a singular boundary-layer problem in the continuum into a regular perturbation problem. However, the continuum limit of the discrete problem is extremely nontrivial and is not completely understood. This article examines two singular boundary-layer problems taken from mathematical physics, the instanton problem and the Blasius equation, and in each case examines two strategies, Padé resummation and variational perturbation theory, to recover the solution to the continuum problem from the solution to the associated discrete problem. Both resummation procedures produce good and interesting results for the two cases, but the results still deviate from the exact solutions. To understand the discrepancy a comprehensive large-order behavior analysis of the strong-coupling lattice expansions for each of the two problems is done. © 2002 American Institute of Physics. [DOI: 10.1063/1.1490408]

I. INTRODUCTION

In this article we report some major advances in understanding (albeit not a complete solution to) a difficult general class of problems in mathematical physics. We consider here the conversion of a continuum problem into a discrete problem by the insertion of a lattice spacing parameter a , the solution of the continuum problem on the lattice, and the subsequent extremely subtle continuum limit $a \rightarrow 0$.

Almost every continuum physics problem is singular as a function of the parameters in the problem. As a result, only rarely does the perturbation series take the form of a Taylor series having a nonzero radius of convergence. As an elementary example, consider the algebraic polynomial equation

$$\epsilon x^3 + x - 1 = 0. \quad (1)$$

This problem is singular in the limit $\epsilon \rightarrow 0$. In this limit, the degree of the polynomial changes from three to one and thus two of the roots abruptly disappear. As a consequence, a perturbative solution to this problem [expressing the roots $x(\epsilon)$ as series in powers of ϵ] yields expressions that are more complicated than Taylor series.

A more elaborate example of a singular problem is the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2M} \nabla^2 \Psi(\mathbf{x}) + [V(\mathbf{x}) - E] \Psi(\mathbf{x}) = 0. \quad (2)$$

^{a)}Electronic mail: cmb@howdy.wustl.edu

^{b)}Electronic mail: pelster@physik.fu-berlin.de

^{c)}Electronic mail: weisbach@physik.fu-berlin.de

In the classical limit $\hbar \rightarrow 0$ this differential equation abruptly becomes an *algebraic* equation, and thus the general solution no longer contains any arbitrary constants or functions and, as a result, it can no longer satisfy the initial conditions. We know that for small $\hbar$ the solution is not Taylor-like but rather is a singular exponential in WKB form:

$$\Psi(\mathbf{x}) \sim e^{S(\mathbf{x})/\hbar} \quad (\hbar \rightarrow 0). \quad (3)$$

In the study of quantum field theory, it is well known that infinities appear in the perturbative expansion in powers of the coupling constant. There are two kinds of infinities. The first kind, which is due to the pointlike nature of the interaction, requires the use of renormalization. The second kind, which is due to singularities in the complex-coupling-constant plane, forces the perturbation series to have a zero radius of convergence.

A quantum field theory can be regulated by introducing a lattice spacing. The resulting discrete theory is completely finite and can be studied numerically by using various kinds of numerical methods such as Monte Carlo integration. However, the underlying singular nature of the continuum quantum field theory resurfaces in the continuum limit $a \rightarrow 0$. The introduction of a lattice spacing and the singular nature of the continuum limit was investigated in a series of papers by Bender *et al.*¹⁻⁹

A quantum field theory is just one instance in which discretization regulates and eliminates the singular nature of the problem. It is also known that introducing a lattice spacing converts a boundary-layer problem, which is a singular perturbation problem, into a regular perturbation problem.¹⁰⁻¹² A *boundary-layer problem* is a differential-equation-boundary-value problem in which the highest derivative of the differential equation is multiplied by a small parameter ϵ . Consider as an example

$$\epsilon y''(x) + a(x)y'(x) + b(x)y(x) = c(x), \quad (4)$$

where the boundary conditions on the function $y(x)$ typically have a form such as

$$y(0) = A, \quad y(1) = B. \quad (5)$$

This boundary-value problem is singular because in the limit $\epsilon \rightarrow 0$ one of the solutions abruptly disappears and the limiting solution is not able to satisfy the two boundary conditions in (5). The usual way to solve the boundary-value problem (4) and (5) is to decompose the interval $0 \leq x \leq 1$ into two regions, an *outer region*, in which the solution varies slowly as a function of x , and an *inner region* or *boundary-layer region*, in which the solution varies rapidly as a function of x . The boundary-layer region is a narrow region whose thickness is typically of order ϵ or some power of ϵ .¹³

An important example of a boundary-layer problem is the instanton equation

$$\epsilon^2 f''(x) + f(x) - f^3(x) = 0, \quad (6)$$

with the associated boundary conditions

$$f(0) = 0, \quad f(\infty) = 1. \quad (7)$$

The exact solution to this instanton problem is

$$f(x) = \tanh \frac{x}{\epsilon\sqrt{2}}. \quad (8)$$

Note that the solution $f(x)$ varies rapidly at the origin $x = 0$ over a region of thickness ϵ ; this is the boundary-layer region. The solution varies slowly (it is approximately 1) outside of this region. The outer region consists of those x not near the origin.

A novel way to solve the instanton problem is to discretize it by introducing a lattice. On the lattice, the differential equation becomes a difference equation that can easily be solved perturbatively. In the continuum limit, as the lattice spacing vanishes, we then obtain a strong-coupling expansion that must be evaluated by means of a Padé or a variational perturbation theory method. To illustrate the approach our objective will be to calculate the slope of the instanton at $x=0$, which from (8) has the value

$$f'(0) = \frac{1}{\epsilon\sqrt{2}}. \quad (9)$$

We introduce a lattice with lattice spacing a so that the real axis is discretized in steps of width a . The spatial coordinate reads $x_n = na$, where the function $f(x)$ assumes the value $f_n = f(x_n)$. On the lattice the second spatial derivative in (6) becomes

$$f''(x) \rightarrow \frac{f_{n+1} - 2f_n + f_{n-1}}{a^2}. \quad (10)$$

Thus, from the instanton equation (6) we obtain the difference equation

$$\frac{\epsilon^2}{a^2}(f_{n+1} - 2f_n + f_{n-1}) + f_n - f_n^3 = 0, \quad (11)$$

where the boundary values follow from (7):

$$f_0 = 0, \quad f_\infty = 1. \quad (12)$$

The natural expansion parameter now is ϵ^2/a^2 , to which we assign the name δ :

$$\delta \equiv \frac{\epsilon^2}{a^2}. \quad (13)$$

The singular perturbation problem in the continuum [whose solution $f(x)$ in (8) does not possess a Taylor expansion in powers of ϵ] has become a *regular* perturbation problem. That is, we can now expand the solution f_n to the difference equation (11) as a *Taylor series* in powers of δ :

$$f_n = a_{n,0} + a_{n,1}\delta + a_{n,2}\delta^2 + \dots \quad (14)$$

We impose the boundary values (12) by requiring that

$$a_{0,0} \equiv 0 \quad \text{and} \quad a_{n,0} \equiv 1 \quad (n \geq 1). \quad (15)$$

Inserting the *ansatz* (14) into the difference equation (11), we get the recursion relation¹⁰

$$a_{n,j} = \frac{1}{2}a_{n+1,j-1} + a_{n,j-1} + \frac{1}{2}a_{n-1,j-1} - \sum_{k=1}^{j-1} a_{n,k}a_{n,j-k} - \frac{1}{2} \sum_{k=1}^{j-1} \sum_{l=1}^{j-k} a_{n,k}a_{n,l}a_{n,j-k-l}. \quad (16)$$

For the first derivative at the origin $x=0$ this leads to the series

$$f'(0) = \lim_{a \rightarrow 0} \frac{f_1 - f_0}{a} = \lim_{a \rightarrow 0} \frac{f_1}{a} = \lim_{a \rightarrow 0} \frac{1}{a} \sum_{j=0}^{\infty} a_{1,j} \delta^j = \lim_{a \rightarrow 0} \frac{1}{a} \left(1 - \frac{\delta}{2} + \frac{\delta^2}{8} + \frac{11\delta^4}{128} + \dots \right). \quad (17)$$

We have calculated the coefficients $a_{1,j}$ with the help of Maple V R7 up to order $j=200$. The first 20 numbers are given in Table I. A complete list of these coefficients can be found on the

TABLE I. The first 20 weak-coupling coefficients $a_{1,j}$ for the instanton problem (15) and (16).

j	$a_{1,j}$	j	$a_{1,j}$
1	$-\frac{1}{8}$	11	$-\frac{2887747}{262144}$
2	$\frac{1}{8}$	12	$\frac{99392471}{4194304}$
3	0	13	$-\frac{215798295}{4194304}$
4	$\frac{11}{128}$	14	$\frac{3781670831}{33554432}$
5	$-\frac{23}{128}$	15	$-\frac{8349041385}{33554432}$
6	$\frac{295}{1024}$	16	$\frac{1188129285795}{2147483648}$
7	$-\frac{589}{1024}$	17	$-\frac{2659104132291}{2147483648}$
8	$\frac{39203}{32768}$	18	$\frac{47890245452569}{17179869184}$
9	$-\frac{80723}{32768}$	19	$-\frac{108383753179167}{17179869184}$
10	$\frac{1354949}{262144}$	20	$\frac{39433620359113981}{274877906944}$

webpage of the author FW.¹⁴ Note that the expansion parameter δ in (17) is not small but rather tends to infinity in the limit as the lattice spacing a approaches zero. Using the parameter δ defined in (13) we rewrite the series (17) as

$$f'(0) = \frac{1}{\epsilon} \lim_{\delta \rightarrow \infty} \sqrt{\delta} \left(1 - \frac{\delta}{2} + \frac{\delta^2}{8} + \frac{11\delta^4}{128} + \cdots \right). \quad (18)$$

Taking into account the exact result (9), we obtain the identity

$$\frac{1}{\sqrt{2}} = \lim_{\delta \rightarrow \infty} \sqrt{\delta} \left(1 - \frac{\delta}{2} + \frac{\delta^2}{8} + \frac{11\delta^4}{128} + \cdots \right). \quad (19)$$

The purpose of this article is to examine equations like (19). This equation shows that the singular nature of the instanton problem has resurfaced in the continuum limit $\delta \rightarrow \infty$ of the lattice expansion. The expression on the right side of (19) should have the value $1/\sqrt{2} = 0.7071067812\dots$, but it is not at all obvious why this is so, and the objective of this article is to analyze this difficult and subtle limit.

This article is organized as follows. In Sec. II we use Padé techniques to perform the limit in (19). We will see that while the results are not bad (the accuracy is about 1%), better methods are needed. We perform the Padé analysis to much higher order than has ever been done before and we discover a new qualitative behavior that has not yet been observed. In Sec. III we try the use of the variational perturbation theory techniques introduced by Kleinert to perform the sum in (19). These techniques increase the accuracy by a factor of about 10, but they still do not give the exact result. While variational perturbation theory works very well in summing the strong-coupling series for the ground-state energy of the anharmonic oscillator,¹⁵ and for the critical exponents of second-order phase transitions,¹⁶ we show that the series in (19) is at the very edge of validity for Kleinert's methods. We then examine the large-order behavior of the terms of the

sum in (19) in Sec. IV. We show definitively that the Taylor expansion has a nonzero radius of convergence and, thus, on the lattice, the instanton problem is a regular perturbation problem.

In Sec. V we turn to a more difficult singular perturbation problem, namely, the Blasius equation of fluid dynamics. We use the same approach as for the instanton equation. In Secs. VI–VIII we study the summation of the lattice perturbation expansion using Padé and variational methods and we examine the large-order behavior of the lattice perturbation series. We find that Padé methods give good but not excellent results and that variational perturbation theory is better than Padé. Again, the series we need to evaluate in the continuum limit lies at the very edge of validity for Kleinert's methods. We also find that, unlike the lattice perturbation expansion coefficients for the instanton problem, the sign pattern of the Blasius weak-coupling series does not alternate. Rather, it is governed by a cosine function with a frequency different from π .

II. PADÉ RESUMMATION FOR THE INSTANTON EQUATION

In this section we examine what happens if we attempt to evaluate the right side of (19) by using Padé techniques. Padé resummation has already been applied to the instanton problem up to 50th order.¹⁰ However, we have been able to perform the procedures to much higher orders. We have discovered that remarkable and unsuspected new phenomena occur just a few orders beyond what has been computed.

The procedure is as follows. Consider the formal Frobenius series

$$S(\delta) = \delta^M \sum_{n=0}^{\infty} a_n \delta^n, \quad (20)$$

where M is a non-negative number. Raising this series to the power $1/M$, inverting the right hand side and reexpanding, we obtain

$$S^{1/M}(\delta) = \frac{\delta}{\sum_{n=0}^{\infty} b_n \delta^n}, \quad (21)$$

with new expansion coefficients b_n . Assuming we know the first $N+1$ terms of the original power series in (20), we raise Eq. (21) to the power N . We then truncate the summation at $n = N$, finally getting

$$S^{N/M}(\delta) = \frac{\delta^N}{\sum_{n=0}^N c_n^{(N)} \delta^n}, \quad (22)$$

where we have reexpanded and obtained new expansion coefficients c_n . In the limit $\delta \rightarrow \infty$, only the N th term in the denominator survives and we obtain the approximant

$$(S_N)^{N/M} \equiv \lim_{\delta \rightarrow \infty} S^{N/M}(\delta) = \lim_{\delta \rightarrow \infty} \frac{\delta^N}{\sum_{n=0}^N c_n^{(N)} \delta^n} = \frac{1}{c_N^{(N)}}. \quad (23)$$

The approximant $S_N = (c_N^{(N)})^{-M/N}$ is the zeroth-order survivor of the limiting process. Also, taking into account the first-order correction we observe that, as in the case of variational perturbation theory (see Sec. III), there is an approach to scaling. In the limit $\delta \rightarrow \infty$ the Frobenius series $S(\delta)$ in Eq. (20) converges to a constant C . Additionally, the approach to scaling, following from the Padé resummation (23), reveals how fast it converges:

$$S(\delta) \sim C + C' \delta^{-1} \quad (\delta \rightarrow \infty). \quad (24)$$

We now apply this procedure to the boundary-layer problem (11). [Recall that the weak-coupling coefficients for the first 20 coefficients $a_{1,j}$ obtained from (16) are shown in Table I and that more can be found in Ref. 14.] Resumming the series (14) for $n = 1$,

TABLE II. The first 20 Padé approximants for the solution to the instanton problem (19).

N	S_N	N	S_N
1	1	11	0.709 998 411
2	0.840 896 415	12	0.708 235 422
3	0.781 934 407	13	0.706 789 935
4	0.757 237 797	14	0.705 659 505
5	0.740 759 114	15	0.704 734 605
6	0.731 210 449	16	0.704 006 945
7	0.723 927 185	17	0.703 419 862
8	0.719 045 188	18	0.702 964 717
9	0.715 146 335	19	0.702 610 220
10	0.712 308 458	20	0.702 349 024

$$f_1 = \sum_{j=0}^N a_{1,j} \delta^j, \quad (25)$$

according to the Padé procedure (23) with $M = \frac{1}{2}$ as follows from (19) and evaluating the approximants $S_N = (c_N^{(N)})^{-M/N}$, we get the numbers listed in Table II.

Compared with the numerical solution $1/\sqrt{2} \approx 0.717\,106\,781\,2$, this strong-coupling expansion seems to converge quite well. However, when we go to higher orders, we find that the numbers drop below the exact solution and assume a minimum at $N=24$, where the approximant has the value $S_{24} \approx 0.701\,983\,19$. The approximants then rise again, cross the exact solution at $N=41$ and become complex at $N=52$. The appearance of complex numbers is a consequence of taking the N th root in Eq. (23) when the coefficients $c_N^{(N)}$ become negative. This phenomenon has not been observed before in the course of using this Padé procedure. The imaginary part then becomes smaller and smaller as N rises. Abruptly, at $N=68$, the approximants become real again. As one can see from the spikes in Fig. 1 this pattern is repeated for higher N . Note that the figure only shows the real part of the Padé approximant S_N .

Apparently, the sequence of approximants S_N does not converge. The singular nature of the instanton equation has the effect of making the Padé approximants behave like the partial sums of a divergent (asymptotic) series; at first the partial sums appear to converge to a limit, and then they veer off. In the case of the Padé's shown in Fig. 1 the approximants approach to within 1% of the

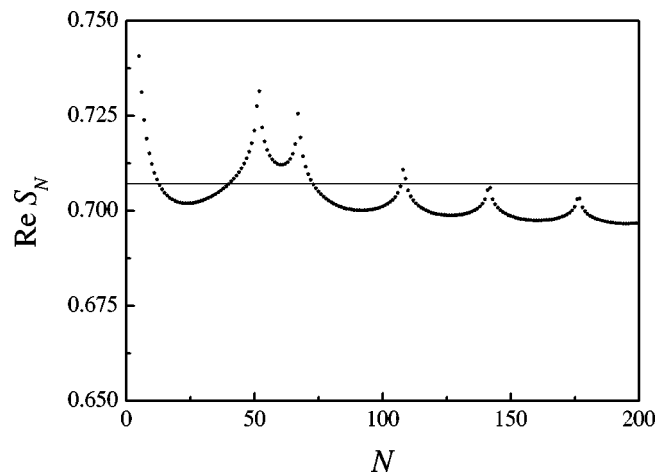


FIG. 1. The real part of the Padé approximants S_N up to 200th order. Note that the approximants do not converge to the exact solution, which is represented by the horizontal solid line. The phases where the approximants become complex are marked by spikes.

correct limit before veering off. It appears that another more powerful resummation technique is needed to treat the expression in (19). In the next section we apply a technique due to Kleinert.

III. VARIATIONAL PERTURBATION THEORY FOR THE INSTANTON EQUATION

Kleinert has developed a technique in the context of the ground-state energy of the anharmonic oscillator¹⁵ and of critical exponents of second-order phase transitions¹⁶ for summing divergent perturbation series. This technique, known as Kleinert's square-root trick, is described below.

Consider a weak-coupling series

$$f_N(\delta) = \sum_{n=0}^N f_n \delta^n, \quad (26)$$

which is truncated at order N . Rewrite this weak-coupling expansion by introducing an auxiliary scaling parameter κ .^{15,16}

$$f_N(\delta) = \kappa^p \sum_{n=0}^N f_n \left(\frac{\delta}{\kappa^q} \right)^n \bigg|_{\kappa=1}, \quad (27)$$

which is set to $\kappa=1$ later. The square-root trick now reads

$$\kappa \rightarrow \sqrt{K^2 + \kappa^2 - K^2} = K \sqrt{1 + \delta r}, \quad (28)$$

where K is a “dummy” scaling parameter and

$$r = \frac{1}{\delta} \left(\frac{\kappa^2}{K^2} - 1 \right). \quad (29)$$

In the case of the anharmonic oscillator, K is the frequency Ω of a trial harmonic oscillator.¹⁵

Substituting (28) into the truncated weak-coupling series (27), we obtain

$$f_N(\delta) = \sum_{n=0}^N f_n K^{p-nq} (1 + \delta r)^{(p-nq)/2} \delta^n. \quad (30)$$

The factor $(1 + \delta r)^\alpha$ with $\alpha \equiv (p - nq)/2$ is expanded by means of generalized binomials:

$$(1 + \delta r)^\alpha = \sum_{k=0}^{N-n} \binom{\alpha}{k} (\delta r)^k \mathcal{O}(\delta^{N-n}) = \sum_{k=0}^{N-n} \binom{\alpha}{k} \left(\frac{1}{K^2} - 1 \right)^k \mathcal{O}(\delta^{N-n}), \quad (31)$$

where we have used (29) and finally have set $\kappa \equiv 1$. The binomial is defined as

$$\binom{\alpha}{k} \equiv \frac{\Gamma(\alpha + 1)}{\Gamma(k + 1) \Gamma(\alpha - k + 1)}. \quad (32)$$

Thus the function $f_N(\delta)$ becomes K dependent and reduces to

$$f_N(\delta, K) = \sum_{n=0}^N \left[\sum_{k=0}^{N-n} \binom{\frac{1}{2}(p-nq)}{k} \left(\frac{1}{K^2} - 1 \right)^k K^{p-nq} \right] f_n \delta^n. \quad (33)$$

To first order this expression reduces to

$$f_1(\delta, K) = \left(1 - \frac{p}{2}\right) f_0 K^p + \frac{p}{2} f_0 K^{p-2} + f_1 \delta K^{p-q}. \quad (34)$$

Applying the principle of least sensitivity¹⁷ leaves us with

$$\frac{\partial f_1(\delta, K)}{\partial K} \sim p \left(1 - \frac{p}{2}\right) f_0 + \frac{p(p-2)}{2} f_0 K^{-2} + (p-q) f_1 \delta K^{-q} \equiv 0. \quad (35)$$

Next, making the strong-coupling ansatz

$$K^{(1)}(\delta) = \delta^{1/q} (k_0^{(1)} + k_1^{(1)} \delta^{-2/q} + \dots), \quad (36)$$

we obtain the following equation from (35):

$$p \left(1 - \frac{p}{2}\right) f_0 + \frac{p(p-2)}{2} f_0 (k_0^{(1)} \delta^{1/q})^{-2} + (p-q) f_1 \delta (\delta^{1/q} k_0^{(1)})^{-q} = 0. \quad (37)$$

The second term is a subleading contribution in the limit as the coupling δ goes to infinity which we can neglect. Solving for $k_0^{(1)}$ we then get

$$k_0^{(1)} = \left(\frac{2 f_1}{f_0} \frac{p-q}{p(p-2)} \right)^{1/q}. \quad (38)$$

Assuming that the ansatz (36) for the variational parameter $K(\delta)$ also holds for higher orders we obtain from the function $f_N(\delta, K)$ in (33)

$$f_N(\delta) = \delta^{p/q} [b_0^{(N)} (k_0^{(N)}) + b_1^{(N)} (k_0^{(N)}, k_1^{(N)}) \delta^{-2/q} + \dots], \quad (39)$$

where the leading strong-coupling coefficient $b_0^{(N)}(k_0^{(N)})$ is given by

$$b_0^{(N)}(k_0^{(N)}) = \sum_{n=0}^N \sum_{k=0}^{N-n} \binom{\frac{1}{2}(p-nq)}{k} (-1)^k f_n(k_0^{(N)})^{p-nq}. \quad (40)$$

The inner sum can be further simplified, using

$$\sum_{k=0}^m (-1)^k \binom{\alpha}{k} = (-1)^m \binom{\alpha-1}{m}. \quad (41)$$

Thus the strong-coupling coefficient (40) reduces to

$$b_0^{(N)}(k_0^{(N)}) = \sum_{n=0}^N (-1)^{N-n} \binom{\frac{1}{2}(p-nq)-1}{N-n} f_n(k_0^{(N)})^{p-nq}. \quad (42)$$

So, looking at Eq. (39) we see that the fraction p/q tells us the leading power behavior in δ and $2/q$ indicates the approach to scaling:

$$\sum_{j=0}^{\infty} f_j \delta^j \sim \delta^{p/q} (b_0 + b_1 \delta^{-2/q} + \dots) \quad (\delta \rightarrow \infty). \quad (43)$$

For the instanton equation we can determine the numbers p and q by re-obtaining the differential equation (6) from the difference equation (11). The positive real axis is discretized in steps of width a , so that we let $x_n \equiv na$. The power series expansion for the discrete function $f_n = f(x_n)$ has the form

$$f_{n\pm 1} = f(x_n) \pm f'(x_n)a + \frac{1}{2}f''(x_n)a^2 \pm \frac{1}{6}f'''(x_n)a^3 + \frac{1}{24}f^{(4)}(x_n)a^4 \pm \dots \quad (44)$$

Thus, the numerator of the second derivative (10) becomes

$$f_{n+1} - 2f_n + f_{n-1} = f''_n a^2 + \frac{1}{12}f^{(4)}_n a^4 + \dots, \quad (45)$$

so the zeroth-, first-, and third-order contributions cancel. Translating the lattice result for f_n back to the continuous function $f(x_n) = f_n$, the difference equation (11) reads

$$\epsilon^2[f''(x) + \frac{1}{12}f^{(4)}(x)a^2 + \dots] + f(x) - f^3(x) = 0. \quad (46)$$

Writing out the power series

$$f(x) = f_0(x) + a^2 f_1(x) + a^4 f_2(x) + \dots, \quad (47)$$

and comparing even powers of a , we get from Eq. (46) for a^0

$$\epsilon^2 f''_0(x) + f_0(x) - f_0^3(x) = 0, \quad (48)$$

which is just the original instanton equation (6). For a^2 we have

$$\epsilon^2 f''_1(x) + f_1(x)(1 - 3f_0^2(x)) = -\frac{1}{12}\epsilon^2 f^{(4)}_0(x). \quad (49)$$

The boundary values read

$$f_0(0) = 0, \quad f_0(\infty) = 1, \quad (50)$$

and

$$f_1(0) = f_1(\infty) = 0, \quad (51)$$

respectively. The solution to Eq. (48) with the boundary values (50) is of course

$$f_0(x) = \tanh \frac{x}{\epsilon\sqrt{2}}. \quad (52)$$

So, finally from (47) we get for the derivative at the origin $x=0$:

$$f'(0) = f'_0(0) + \frac{\epsilon^2}{\delta} f'_1(0) + \dots = \frac{1}{\epsilon\sqrt{2}} + \frac{\epsilon^2}{\delta} f'_1(0) + \dots \quad (53)$$

Comparing Eq. (53) with (18), we resum the weak-coupling series in (18) as

$$1 - \frac{\delta}{2} + \frac{\delta^2}{8} + \dots = \delta^{-1/2} \left[\frac{1}{\sqrt{2}} + \epsilon^3 f'_1(0) \delta^{-1} + \dots \right]. \quad (54)$$

Also, comparing with (43), we conclude that the leading power and the approach to scaling are given by

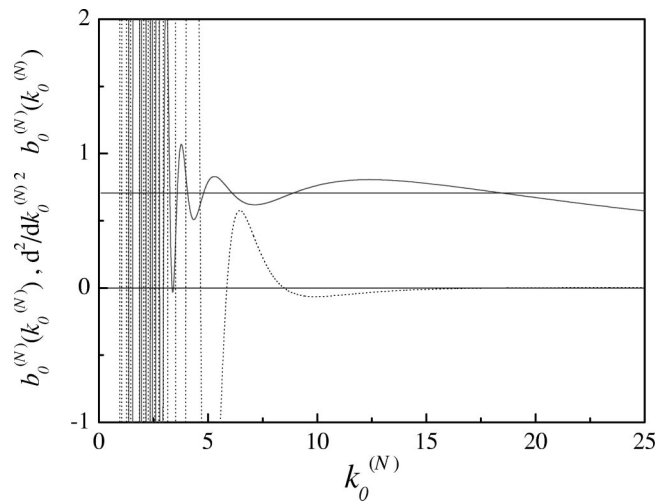


FIG. 2. The function $b_0^{(N)}(k_0^{(N)})$ from (42) for $N=200$ (solid line) and its second derivative with respect to $k_0^{(N)}$ (dotted line). The upper horizontal line equals $1/\sqrt{2}$, the correct limiting value of the instanton problem. All extrema of $b_0^{(N)}$ are far from this value. Only the inflection point on the right-hand side comes close. The value for $k_0^{(N)}$, for which the second derivative vanishes, is $k_0^{(N)} = 18.425\ 10$. Substituting that number into the function $b_0^{(N)}(k_0^{(N)})$, we obtain in the 200th order $b_0^{(200)} = 0.707\ 417$. The corresponding Richardson extrapolations can be found in Table IV.

$$\frac{p}{q} = -\frac{1}{2}, \quad \frac{2}{q} = 1, \quad (55)$$

respectively. So we identify $p = -1$ and $q = 2$.

We now evaluate the leading strong-coupling coefficient b_0 from (43) according to (42) with $p = -1$ and $q = 2$. To that end we substitute our 200 weak-coupling coefficients from Ref. 14 into the formula using a computer algebra program. We are now confronted with the following problem: The principle of least sensitivity cannot be unambiguously applied. Optimizing with respect to extrema, inflection points, or higher derivatives does yield converging results for the strong-coupling limit. However, all these strong-coupling series converge to the wrong values.

There is one particularly unpleasant case: The second derivative with respect to $k_0^{(N)}$ for the largest $k_0^{(N)}$ where this derivative exists (see Fig. 2) gives a convergent strong-coupling series. The numbers come extremely close to $1/\sqrt{2}$ as one can see from the 20 numbers in Table III. The 200th leading strong-coupling coefficient is $b_0^{(200)} = 0.707\ 417\ \dots$. However, a Richardson extrapolation¹³ based on the first 200 orders then unfortunately shows that variational perturbation theory produces a value slightly smaller than $1/\sqrt{2}$. The first six orders of Richardson extrapolation

TABLE III. The last 20 variational strong-coupling coefficients $b_0^{(N)}$ from Eq. (42).

N	$b_0^{(N)}$	N	$b_0^{(N)}$
180	0.707 530 492	190	0.707 471 024
181	0.707 524 250	191	0.707 465 419
182	0.707 518 076	192	0.707 459 872
183	0.707 511 970	193	0.707 454 384
184	0.707 505 930	194	0.707 448 952
185	0.707 499 955	195	0.707 443 575
186	0.707 494 044	196	0.707 438 253
187	0.707 488 197	197	0.707 432 986
188	0.707 482 412	198	0.707 427 771
189	0.707 476 687	199	0.707 422 609

TABLE IV. Six orders of Richardson extrapolations for the strong-coupling coefficient $b_0^{(N)}(k_0^{(N)})$ up to $N=200$ for the instanton problem. The last value is only 0.099% away from the correct limiting value $1/\sqrt{2} = 0.707\,106\,781\,2\dots$

Order	Value for $b_0^{(N)}$	Convergence
1	0.706 400 49	decreasing
2	0.706 399 832 00	increasing
3	0.706 399 832 082	increasing
4	0.706 399 832 085 865 8	increasing
5	0.706 399 832 085 884 411	increasing
6	0.706 399 832 085 884 464 98	increasing

tions are presented in Table IV. Hence, the strong-coupling series $b_0^{(N)}$ does converge, but it converges to the wrong number, only one part per 1000 away from the true value:

$$f_1^{(\text{VPT})} \approx \lim_{\delta \rightarrow \infty} \sum_{n=0}^{200} a_{1,n} \delta^n = b_0^{(\infty)} = 0.706\,399\,832\,085\,88\,45 \pm 0.000\,000\,000\,000\,000\,1 \quad (56)$$

compared with $f'(0) = 1/\sqrt{2} = 0.707\,106\,781\,2\dots$. The deviation is just 0.099%, but $1/\sqrt{2}$ can unfortunately be ruled out.

Given that $p = -1$ and $q = 2$, the failure of variational perturbation theory is not surprising. According to Ref. 16 the fraction $2/q$ must lie within the open interval $(\frac{1}{2}, 1)$. Otherwise, one cannot prove that variational perturbation theory converges. Thus, *this problem lies exactly on the upper boundary of the region in which the summation method is known to work.*

We can understand the upper edge of the range of the parameter $2/q$ that describes the approach to scaling by looking at the standard deviation from the actual limiting value. It turns out¹⁶ that the deviation in the limit as the perturbative order N goes to infinity assumes the shape

$$\left| \frac{b_0^{(N)} - b_0}{b_0} \right| \sim \exp(-CN^{1-2/q}) \quad (N \rightarrow \infty), \quad (57)$$

where C is a constant. So, to obtain exponential convergence for the sequence formed by the $b_0^{(N)}$, we need $1 - 2/q > 0$. In other words, the approach to scaling $2/q$ is bounded and it must be smaller than one. The lower edge is more subtle and is discussed in Ref. 16.

In conclusion, we have applied variational perturbation theory to a case that lies at the very edge of its applicability. We see that variational perturbation theory gives better results by about a factor of 10 than the Padé approximations examined in Sec. II. However, we have not yet found a systematic method for resumming (19) that enables us to perform the continuum limit of the discrete lattice theory. Therefore, we now lay the foundation for further investigations by analyzing the large-order behavior of the instanton series.

IV. LARGE-ORDER BEHAVIOR FOR THE INSTANTON EQUATION

It can be seen from the numerical results in Ref. 14 that the instanton weak-coupling series is of Borel type. That is, it exhibits an alternating sign pattern. From the ratio test we can see that the coefficients $a_{n,j}$ do not grow factorially fast. The large-order behavior of $a_{n,j}$ has the general form

$$a_{n,j} \sim (-1)^{n+j+1} K_n^j A_n B_n \quad (j \rightarrow \infty). \quad (58)$$

The constant A_n can be obtained by evaluating the limit

$$A_n = \lim_{j \rightarrow \infty} \frac{\log[a_{n,j+2} a_{n,j} / (a_{n,j+1})^2]}{\log[j(j+2)/(j+1)^2]}, \quad (59)$$

TABLE V. Six orders of Richardson extrapolations for the exponent A of the large-order instanton weak-coupling coefficients, based on the first 200 weak-coupling coefficients. The value $A = -\frac{3}{2}$ is quite plausible.

Order	Value for A	Convergence
1	-1.4998	increasing
2	-1.500 017	decreasing
3	-1.500 001 1	decreasing
4	-1.499 998 74	increasing
5	-1.500 000 4	decreasing
6	-1.499 999 893	increasing

and the reciprocal of the radius of convergence is

$$K_n = - \lim_{j \rightarrow \infty} \frac{a_{n,j+1}}{a_{n,j}} \left(\frac{j}{j+1} \right)^{A_n}. \quad (60)$$

Also, the overall factor B_n is determined from

$$B_n = \lim_{j \rightarrow \infty} \frac{|a_{n,j}|}{K_n^j j^{A_n}}. \quad (61)$$

Using the 200 weak-coupling coefficients, we find that the exponent A_n and the reciprocal radius of convergence K_n are independent of n . The value of $K_2 = 2.466\,829\,06$ coincides with $K_1 = 2.466\,829\,06$ for all significant digits. The same is true for $A_1 = -1.500\,000$ and $A_2 = -1.500\,000$. Thus, it appears that we may omit the subscripts n for K_n and A_n . In contrast, the data suggest that B_n strongly depends on n . B_n is the numerical value associated with the largest uncertainty. In fact, Eq. (61) suggests that small deviations in K and A lead to dramatic changes in the value of B_n . We calculated A , K , B_1 , and B_2 up to 200th order with the help of Maple V R7. We then extrapolated these 200 orders to infinity using Richardson extrapolation.¹³ We obtained

$$\begin{aligned} A &= -1.500\,000 \pm 0.000\,001, \\ K &= 2.466\,829\,06 \pm 0.000\,000\,1, \\ B_1 &= 0.0171 \pm 0.0001, \\ B_2 &= 0.1190 \pm 0.0001. \end{aligned} \quad (62)$$

Detailed numerical results for the first six Richardson extrapolations for the exponent A , the inverse radius of convergence K , and the overall factors B_1 and B_2 can be found in Tables

TABLE VI. Six orders of Richardson extrapolations for the inverse radius of convergence K of the large-order instanton weak-coupling coefficients, based on the first 200 weak-coupling coefficients under the assumption that $A = -\frac{3}{2}$.

Order	Value for K	Convergence
1	2.466 92	decreasing
2	2.466 828 3	increasing
3	2.466 829 11	decreasing
4	2.466 829 065	decreasing
5	2.466 829 059 7	increasing
6	2.466 829 063 5	decreasing

TABLE VII. Six orders of Richardson extrapolations for the overall factor B_1 of the large-order instanton weak-coupling coefficients, based on the first 200 weak-coupling coefficients under the assumption that $K=2.448\,290\,6$ and $A=-\frac{3}{2}$. The value of B_1 strongly depends on the numerical values for A and K . Changing K in the sixth decimal place influences the third significant figure of B_1 . Also, all the Richardson extrapolations are increasing, so, strictly speaking, we only have a lower boundary for B_1 . Thus, the accuracy of B_1 may not be very good.

Order	Value for B_1	Convergence
1	0.017 083 7	increasing
2	0.017 086 4	increasing
3	0.017 087	increasing
4	0.017 089 3	increasing
5	0.017 090 8	increasing
6	0.017 092 2	increasing

V–VIII. The calculation of B_1 is extremely delicate; changing the inverse radius of convergence in the sixth decimal place influences the third significant figure of B_1 . The same is true of B_2 .

Unfortunately, there is no way to derive these values by applying asymptotic analysis to the recursion relation (16). The problem is that the double summation in this equation includes small j , so we cannot let j go to infinity and use the large-order behavior (58). Substituting the *ansatz* (58) into Eq. (16) and taking the limit leads to contradictory results. For $n=1$ we get

$$Kj^AB_1 = \frac{1}{2}(j-1)^AB_2 + (j-1)^AB_1 - \frac{3}{2}B_1^2K \sum_{k=1}^{j-1} k^A(j-k)^A - \frac{1}{2}B_1^3K \sum_{k=1}^{j-1} \sum_{l=1}^{j-k} k^Al^A(j-k-l)^A. \quad (63)$$

Pulling out some factors and letting $x \equiv k/j$, we obtain for the first summation

$$\lim_{j \rightarrow \infty} \sum_{k=1}^j \left(\frac{k}{j}\right)^A \left(1 - \frac{k}{j}\right)^A = \int_0^1 dx [x(1-x)]^A = \frac{\Gamma^2(A+1)}{\Gamma(2A+2)}, \quad (64)$$

if and only if $A > -1$. For $A < -1$ which is strongly favored by the data we obtain

$$\int_0^1 dx [x(1-x)]^A = 2\zeta(-A). \quad (65)$$

The double summation reduces to

$$\lim_{j \rightarrow \infty} \sum_{k=1}^j \sum_{l=1}^{j-k} \frac{k^Al^A}{j^{2A}} \left(1 - \frac{k}{j} - \frac{l}{j}\right) = \int_0^1 dx \int_0^1 dy [xy(1-x-y)]^A = \frac{\Gamma^3(A+1)}{\Gamma(3A+3)}, \quad (66)$$

where $y \equiv l/j$ and $A > -1$. For $A < -1$ the result is

TABLE VIII. Six orders of Richardson extrapolations for the overall factor B_2 of the large-order instanton weak-coupling coefficients based on the first 200 weak-coupling coefficients and the same assumptions as in the case of B_1 (see Table VII). The value of B_2 depends strongly on A and K .

Order	Value for B_2	Convergence
1	0.119 069	increasing
2	0.119 083	increasing
3	0.119 093	increasing
4	0.119 054 095	increasing
5	0.119 054 125	increasing
6	0.119 054 146	increasing

$$\int_0^1 dx \int_0^1 dy [xy(1-x-y)]^A = 3\zeta^2(-A). \quad (67)$$

Substituting the results in (65) and (67) into (63) leads to a contradiction: The inverse radius of convergence then turns out to be

$$K = \frac{1 + B_2/2B_1}{1 + 3\zeta(\frac{3}{2})B_1 + \frac{3}{2}\zeta^2(\frac{3}{2})B_1^2}, \quad (68)$$

which would imply that, given $B_1 = 0.0171$ and $B_2 = 0.1190$, the value of K would be

$$K = 3.940. \quad (69)$$

This result can be ruled out because of the numerical result (62). Also, (68) does not contain the exponent A because all the factors j^A in (63) cancel. So A cannot be determined analytically using this asymptotic analysis.

V. BOUNDARY LAYERS ON THE LATTICE—BLASIUS EQUATION

The Blasius equation¹⁸ arises in the study of fluid dynamics. It is a special limiting case of the Navier–Stokes equation and determines the flow of an incompressible fluid across a semi-infinite flat plate. The equation reads

$$2\epsilon y'''(x) + y(x)y''(x) = 0. \quad (70)$$

Assuming that the tangential velocity $y'(x)$ at the outer limit of the boundary layer is constant, the boundary conditions read¹⁹

$$y(0) = y'(0) = 0, \quad y'(\infty) = 1. \quad (71)$$

Our objective here is to calculate the second derivative $y''(0)$, which represents the stress on the plate. We discretize the Blasius equation (70) by introducing a lattice spacing a :

$$2\delta(f_{n+1} - 3f_n + 3f_{n-1} - f_{n-2}) + f_n(f_{n+1} - 2f_n + f_{n-1}) = 0, \quad (72)$$

where we define $f_n \equiv y(na)/a$ and $\delta \equiv \epsilon/a^2$. The boundary conditions (71) now read

$$f_0 = f_{-1} = 0, \quad f_n \sim n \quad (n \rightarrow \infty). \quad (73)$$

Expanding f_n as a series in powers of δ as in Eq. (14), we obtain the recursion relation¹⁰

$$\begin{aligned} a_{n+1,j} - 2a_{n,j} + a_{n-1,j} = & -\frac{2}{n}(a_{n+1,j-1} - 3a_{n,j-1} + 3a_{n-1,j-1} - a_{n-2,j-1}) \\ & - \frac{1}{n} \sum_{k=1}^{j-1} a_{n,k}(a_{n+1,j-k} - 2a_{n,j-k} + a_{n-1,j-k}). \end{aligned} \quad (74)$$

The boundary values are

$$\begin{aligned} a_{n,0} &= n \quad (n \geq 0), \\ a_{-1,0} &= 0, \\ a_{-n-1,j} &= a_{n,j} \quad (n \geq 0). \end{aligned} \quad (75)$$

TABLE IX. The first 20 weak-coupling coefficients for the Blasius recursion relation (74) and (75). Observe that the coefficients $a_{1,j}$ are not of Borel type (they do not alternate in sign). A cosine function with a frequency different from π governs the sign pattern (see Sec. VIII).

j	$a_{1,j}$	j	$a_{1,j}$
1	-2	11	$\frac{30868632383}{5457375}$
2	2	12	$\frac{6325029622}{637875}$
3	$\frac{8}{3}$	13	$-\frac{487693745019181}{13408770375}$
4	-6	14	$-\frac{4774319527974167}{37819608750}$
5	$-\frac{184}{15}$	15	$\frac{430321251088745734}{2212447111875}$
6	$\frac{136}{9}$	16	$\frac{796235344548876790517}{603998061541875}$
7	$\frac{11062}{105}$	17	$-\frac{2249988054506764174584049}{6776858250499837500}$
8	$-\frac{8162}{225}$	18	$-\frac{178060537619150189817796}{14237097164915625}$
9	$-\frac{10557416}{14175}$	19	$-\frac{13224896152219729667498038639}{1301909768346024337500}$
10	$-\frac{57628622}{99225}$	20	$\frac{121756993154067534451733120837029}{1153217968487557347375000}$

Equation (74) can be solved order by order by using a computer algebra program. Table IX shows the first 20 weak-coupling coefficients $a_{1,j}$. All coefficients up to the 300th order can be found Ref. 20.

VI. PADÉ RESUMMATION FOR THE BLASIUS EQUATION

We now resum the weak-coupling coefficients using the Padé method (23) with $M = \frac{1}{2}$. This value of M will be derived in Sec. VII in Eq. (82). The exact solution¹⁰ to the Blasius equation (70), obtained numerically up to five digits, is $y''(0) = 0.332\,06$. Unfortunately, the sequence formed by the approximants S_N appears to converge, but not to the correct value. According to Table X the sequence becomes very flat and Richardson extrapolation¹³ shows that the S_N ap-

TABLE X. The first 20 Padé approximants for the solution to the Blasius equation (70). The sequence formed by the S_N converges extremely slowly.

N	S_N	N	S_N
1	0.5	11	0.357 463 212 1
2	0.420 448 207 6	12	0.356 332 665 1
3	0.394 820 183 0	13	0.355 384 804 8
4	0.381 944 373 2	14	0.354 579 594 4
5	0.374 206 230 9	15	0.353 888 284 2
6	0.369 050 481 1	16	0.353 289 150 9
7	0.365 377 967 3	17	0.352 765 581 3
8	0.362 635 906 0	18	0.352 304 658 8
9	0.360 515 591 5	19	0.351 896 192 9
10	0.358 830 970 7	20	0.351 532 039 9

TABLE XI. Three orders of Richardson extrapolations for the Blasius equation (70), based on the first 70 Padé approximants S_N .

Order	Value of $y''(0)$	Convergence
1	0.3445	decreasing
2	0.3436	decreasing
3	0.3430	oscillating

proach the wrong limiting value (see Table XI). A third-order Richardson gives $S_\infty = 0.3430$, based on the first 70 weak-coupling coefficients. This value is significantly higher than the correct value $y''(0) = 0.33206$, the deviation is 3.3%.

The failure of the Padé resummation is not surprising because the Padé method assumes the approach to scaling δ^{-1} according to (24). However, in the case of the Blasius equation the approach to scaling is $\delta^{-1/2}$, as we will see in Eq. (82) in the next section.

VII. VARIATIONAL PERTURBATION THEORY FOR THE BLASIUUS EQUATION

Variational perturbation theory for the Blasius equation fails to converge to the correct answer in the same way as for the instanton problem. We determined the leading strong-coupling term (42) up to 200th order and again it was impossible to find extrema, inflection points, or higher derivatives that yield the correct result. Tables XII and XIII show the last 20 strong-coupling coefficients $b_0^{(N)}$ and six orders of Richardson extrapolation. By determining the values of p and q we show why variational perturbation is likely to fail for this problem.

Consider again the Taylor expansions for $f_{n\pm 1}$ in (44) together with the Taylor series for $f_{n-2} = f(x_n - 2a)$, namely,

$$f_{n-2} = f(x_n) - 2f'(x_n)a + 2f''(x_n)a^2 - \frac{4}{3}f'''(x_n)a^3 + \frac{2}{3}f''''(x_n)a^4 \pm \dots \quad (76)$$

Inserting these expressions into the difference equation for the Blasius problem (72) and translating back to the continuous function $f(x_n) = f_n$, we get

$$2\epsilon(f'''(x)a - \frac{1}{2}f''''(x)a^2 + \dots) + f(x)(f''(x)a^2 + \frac{1}{2}f''''(x)a^4 + \dots) = 0. \quad (77)$$

Next we transform back to the function $y(x) = af(x)$ and assume the Taylor series

$$y(x) = y_0(x) + ay_1(x) + a^2y_2(x) + \dots \quad (78)$$

To zeroth order in a we obtain

TABLE XII. The last 20 variational strong-coupling coefficients $b_0^{(N)}$ for the Blasius equation. The very last coefficient is $b_0^{(200)} = 0.336\,959\,312\,377\,13$, as opposed to the correct value $y''(0) = 0.33206$.

N	$b_0^{(N)}$	N	$b_0^{(N)}$
180	0.336 960 177 930 94	190	0.336 959 711 196 46
181	0.336 960 127 770 85	191	0.336 959 668 491 39
182	0.336 960 078 430 82	192	0.336 959 626 448 43
183	0.336 960 029 893 08	193	0.336 959 585 053 96
184	0.336 959 982 140 34	194	0.336 959 544 294 71
185	0.336 959 935 155 75	195	0.336 959 504 157 74
186	0.336 959 888 922 92	196	0.336 959 464 630 46
187	0.336 959 843 425 91	197	0.336 959 425 700 58
188	0.336 959 798 649 18	198	0.336 959 387 356 12
189	0.336 959 754 577 60	199	0.336 959 349 585 40

TABLE XIII. Six orders of Richardson extrapolations for the strong-coupling coefficient $b_0^{(N)}(k_0^{(N)})$ up to $N=200$ for the Blasius equation. The last value is 1.5% away from the correct limiting value $y''(0)=0.33206$.

Order	Value for $b_0^{(N)}$	Convergence
1	0.336 951 8	increasing
2	0.336 955 563	increasing
3	0.336 955 600 539	increasing
4	0.336 955 600 880 3	increasing
5	0.336 955 600 883 462	increasing
6	0.336 955 600 883 492 32	increasing

$$2\epsilon y_0'''(x) + y_0(x)y_0''(x) = 0, \quad (79)$$

which is just the Blasius equation (70). The small parameter a , which is the lattice spacing, relates ϵ and δ by $a = \sqrt{\epsilon/\delta}$. Thus, if we evaluate the Taylor series (78) for the second derivative at the origin, we see that

$$y''(0) = y_0''(0) + ay_1''(0) + \cdots = \frac{0.332\,06}{\sqrt{\epsilon}} + \sqrt{\frac{\epsilon}{\delta}} y_1''(0) + \cdots. \quad (80)$$

Comparing this series to the original weak-coupling series

$$y''(0) = \sqrt{\frac{\delta}{\epsilon}} (1 - 2\delta + 2\delta^2 + \cdots), \quad (81)$$

we can now determine the leading power p/q and the approach to scaling $2/q$:

$$1 - 2\delta + 2\delta^2 + \cdots = \delta^{-1/2} (0.332\,06 + \delta^{-1/2} \epsilon y_1''(0) + \cdots), \quad (82)$$

so we obtain $p = -2$ and $q = 4$.

Again we find that the approach to scaling $2/q = \frac{1}{2}$ lies just on the boundary of the open interval $(\frac{1}{2}, 1)$, for which the proof of convergence¹⁶ holds. This situation here is the opposite of the instanton case in that it sits at the *lower* boundary of the open interval in which variational perturbation theory works.

VIII. LARGE-ORDER BEHAVIOR FOR THE BLASIUS EQUATION

The Blasius equation exhibits a large-order behavior which is a more subtle than for the instanton problem (58). The Blasius weak-coupling coefficients are not of Borel type; that is, the sign pattern is not alternating. Rather, the sign structure is governed by a cosine function with a frequency that is significantly different from π . Remarkably, it turns out that a pure cosine $\cos(an)$ cannot reproduce all signs correctly. Up to 300th order the sign structure given by $\cos(an)$ is broken twice: The signs at $n=62$ and at $n=212$ are not correct if we optimize with respect to a . So we must consider an additional phase shift $\cos(an+b)$. The parameter b turns out to be slightly smaller than π , but it reproduces all 300 signs correctly.

In order to determine the numerical values of a and b we define

$$f(a, b) \equiv \sum_{n=1}^N \frac{\cos(an+b)}{|\cos(an+b)|} \frac{a_{1,n}}{|a_{1,n}|}. \quad (83)$$

The sum ends at $N=300$ because this is as high as we can calculate using Maple; we know the first 300 weak-coupling coefficients $a_{1,j}$. For the correct values of a and b the function $f(a, b)$

TABLE XIV. Examples of the parameters a and b that give the first 300 signs of the Blasius weak-coupling coefficients correctly, assuming that the sign structure of the underlying large-order behavior is of the form $\cos(an+b)$. The last two values for a can be obtained approximately by summing 2π to the first two values.

a	b
1.3941	3.09
1.3939	3.11
7.67830	3.031
7.67686	3.130

must be equal to 300. We then plot the function $f(a,b)$ over the $a-b$ plane and search for peaks. A careful study of the peaks yields values for a and b which allow the function $f(a,b)$ to assume its maximum at 300. These numbers are given in Table XIV.

The large-order behavior of the Blasius weak-coupling coefficients (unlike the large-order behavior of the instanton coefficients) has an additional overall factor $\cos(an+b)$, and we can now see that the remaining structure differs from the structure of the instanton weak-coupling coefficients. Dividing by the cosine, we observe that the coefficients

$$a'_j \equiv \frac{a_{1,j}}{\cos(aj+b)} \quad (84)$$

grow factorially fast. Thus, we also divide by $j!$:

$$b_j \equiv \frac{a_{1,j}}{\cos(aj+b)j!}. \quad (85)$$

The coefficients b_j are unstable under a ratio test. That is, the ratio b_{j+1}/b_j decreases and then begins to oscillate. This is the inaccuracy that results from the delicate sign pattern of the first 300 coefficients $a_{1,j}$.

ACKNOWLEDGMENTS

C.M.B. is grateful to the U.S. Department of Energy for financial support. F.W. and A.P. wish to thank Hagen Kleinert for fruitful discussions on variational perturbation theory. A.P. acknowledges financial support from the German Research Foundation (DFG) under Contract No. KON 1823/2001.

- ¹C. M. Bender, F. Cooper, G. S. Guralnik, and D. H. Sharp, Phys. Rev. D **19**, 1865 (1979).
- ²C. M. Bender, F. Cooper, G. S. Guralnik, R. Roskies, and D. H. Sharp, Phys. Rev. Lett. **43**, 537 (1979).
- ³C. M. Bender, F. Cooper, G. S. Guralnik, H. Rose, and D. H. Sharp, J. Stat. Phys. **22**, 647 (1980).
- ⁴C. M. Bender, F. Cooper, G. S. Guralnik, R. Roskies, and D. H. Sharp, in *Recent Developments in High-Energy Physics*, edited by B. Kursunoglu, A. Perlmutter, and L. F. Scott (Plenum, New York, 1980), p. 211.
- ⁵C. M. Bender, F. Cooper, R. Kenway, and L. M. Simmons, Jr., Phys. Rev. D **24**, 2693 (1981).
- ⁶C. M. Bender and R. Z. Roskies, Phys. Rev. D **25**, 427 (1982).
- ⁷C. M. Bender, F. Cooper, R. Kenway, and L. M. Simmons, Jr., Phys. Lett. B **109**, 63 (1982).
- ⁸C. M. Bender, L. R. Mead, and L. M. Simmons, Jr., Phys. Rev. D **28**, 936 (1983).
- ⁹C. M. Bender, F. Cooper, and A. Das, Phys. Rev. Lett. **50**, 397 (1983).
- ¹⁰C. M. Bender, F. Cooper, G. S. Guralnik, E. Mjolsness, H. A. Rose, and D. H. Sharp, Adv. Appl. Math. **1**, 22 (1980).
- ¹¹C. M. Bender, Los Alamos Sci. **2**, 76 (1981).
- ¹²C. M. Bender and A. Tovbis, J. Math. Phys. **38**, 3700 (1997).
- ¹³C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw-Hill, New York, 1978).
- ¹⁴The first 200 weak-coupling coefficients for the instanton equation can be found at <http://www.physik.fu-berlin.de/~weissbach/inst.html>
- ¹⁵H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, and Polymers Physics*, 2nd ed. (World Scientific, Singapore, 1995).
- ¹⁶H. Kleinert and V. Schulte-Frohlinde, *Critical Properties of ϕ^4 Theories* (World Scientific, Singapore, 2001).
- ¹⁷P. M. Stevenson, Phys. Rev. D **23**, 2916 (1981).

¹⁸H. Blasius, Z. Math. Phys. **56**, 1 (1908).

¹⁹Z. Belhachmi, B. Brighi, and K. Taous, Acta Math. Univ. Comenianae, **LXIX**, 199 (2000).

²⁰The first 300 weak-coupling coefficients for the Blasius equation can be found at <http://www.physik.fu-berlin.de/~weissbach/blas.html>