

On the generalization of the Duru-Kleinert-propagator transformations

Axel Pelster, Arne Wunderlin

Institut für Theoretische Physik und Synergetik, Universität Stuttgart, Pfaffenwaldring 57, W-7000 Stuttgart 80, Federal Republic of Germany

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Using the concepts of both Schrödinger and Feynman we perform a transformation between different classes of quantum mechanical systems. A carefully elaborated succession of three separate mappings, concerning time, space and the wave function, leads to a relationship between the potentials corresponding to the quantum mechanical systems and their related propagators. Since we admit a space transformation which may depend explicitly on time, our result contains the Duru-Kleinert-propagator transformations as a special case. Additionally we obtain as an example the transformation of the free particle to the harmonic oscillator.

1. Introduction

Quantum mechanics can be formulated on the basis of two fundamentally different concepts which turn out to be equivalent from a mathematical point of view. Originally Schrödinger [1] described microscopic phenomena in terms of a partial differential equation for the wave function. His approach allows to solve important quantum mechanical problems including the hydrogen atom by applying the highly developed techniques for solving partial differential equations. On the other hand the connection of his approach to classical mechanics is not simply obvious. Many years later Feynman [4, 6] succeeded in performing a different, global formulation of quantum mechanics by regarding the so-called propagator as the basic quantity. This approach has been initiated by Dirac [2, 3] and is based on the fundamental notions and concepts of classical mechanics, i.e. the action and the Lagrangian as well as the Hamilton principle. Despite the elegance and the advantages of this method from a pure theoretical point of view, however, it is by no means a trivial task to evaluate the emerging path integrals for the propagator or the energy-dependent Green's function corresponding to many physical problems of practical interest. On the contrary, the math-

ematical difficulties turned out to be so tough that it was, for example, not possible to handle the well-known hydrogen atom using Feynman's original description of quantum mechanics. In fact, the exact analytical expression for the energy-dependent Green's function could only be constructed by using the energy eigenvalues and eigenfunctions which were known from the solution of the Schrödinger equation [5].

Owing to these stimulating difficulties, Kleinert and Duru in 1979 discovered a fertile technique for evaluating propagators or Green's functions [8, 9, 11]. They constructed a powerful transformation method which allows one to map the propagator of a given quantum mechanical system into a new one by performing a reparametrisation of paths with a new pseudo time. When applying this method to the outstanding problem of the hydrogen atom, they could extend the Kustaanheimo-Stiefel transformation of celestial mechanics [7] to quantum mechanics. In this way they mapped the hydrogen atom to the harmonic oscillator and succeeded in calculating the exact Green's function for the hydrogen atom.

This paper aims to present a generalization of the Duru-Kleinert-propagator transformations by performing a time-dependent space transformation and by systematically introducing an additional transformation of the wave function. Our generalization extends the class of propagator transformations between different quantum mechanical systems. Apart from the Duru-Kleinert-propagator transformations we obtain the mapping of the free particle to the harmonic oscillator as a special case. Thus a result is included which was originally found by Inomata et al. [10] but which is excluded by the Duru-Kleinert-propagator transformations. We expect that it might become possible in the near future to discover further and more interesting physical problems which can be treated with our generalization.

The article is organized as follows: In Sect. 2 we review some basic facts concerning the connection between Schrödinger's and Feynman's description of quantum mechanics. We define our notation and summarize im-

portant formulae. The following Sects. 3–5 develop step by step the intended transformation of the propagator in the Lagrangian path integral representation. The transformation is based on the fact that each physical quantity which appears in Schrödinger's equation can be changed by applying a suitable, individual mapping. This observation leads to three well separated mappings, that is of time, of space, and of the wave function. Each of these three independent mappings adds an unknown function which is at our disposal. As we will show in detail, it becomes essential to relate these three functions by two conditions in order to discover the Lagrangian path integral of a new quantum mechanical system after the sequel of all three individual mappings. At the end of Sect. 5 we are able to quote the result which consists of an explicit propagator transformation formula. In Sect. 6 the corresponding mappings are performed independently on the level of the partial differential equation for the propagator to verify our propagator transformation formula. Section 7 discusses two applications. Firstly the Duru-Kleinert-propagator transformations are recovered as one special case. Secondly the transformation of the free particle to the harmonic oscillator is reconsidered from our point of view.

2. Basic facts

Here it is our concern to introduce fundamental notions and results of quantum mechanics and to elaborate the formal connection between the concepts of Schrödinger and Feynman.

In Schrödinger's theory the time evolution of a quantum mechanical system can be formulated as the initial value problem:

$$i\hbar \partial_t |\psi\rangle_t = \hat{H} |\psi\rangle_t \quad \text{with} \quad |\psi\rangle_{t=t_0} = |\psi\rangle_0, \quad (1)$$

where we used Dirac's notation. The state vector $|\psi\rangle_t$ at time t is an element of the underlying Hilbert space $\mathcal{H}_1$ and $\hat{H}$ denotes the time-independent Hamiltonian which defines a closed system. When we specify $\hat{H}$ later on, we will confine ourselves to a particle with mass m which moves in one dimension x with momentum p under the influence of a potential $V=V(x)$:

$$\hat{H} = H(\hat{p}, \hat{x}) = \frac{\hat{p}^2}{2m} + V(\hat{x}). \quad (2)$$

$\hat{x}$ and $\hat{p}$ denote the position and the momentum operator, respectively.

Equation (1) is formally solved by introducing the time-evolution operator $\hat{U}$:

$$|\psi\rangle_t = \hat{U}(t, t_0) |\psi\rangle_0. \quad (3)$$

Because the Hamiltonian $\hat{H}$ is assumed to be time-independent in our case (2), $\hat{U}$ depends only on the time difference $t-t_0$ and has the explicit form:

$$\hat{U}(t, t_0) = \exp \left\{ -\frac{i}{\hbar} \hat{H}(t-t_0) \right\}. \quad (4)$$

The solution (3) can be transformed into its corresponding space representation. Introducing as a complete and orthonormal basis of the Hilbert space $\mathcal{H}_1$ the eigenvectors $|x\rangle$ of the position operator $\hat{x}$ which are defined by

$$\hat{x}|x\rangle = x|x\rangle, \quad (5)$$

we obtain

$$\langle x|\psi\rangle_t = \int dx_0 \langle x|\hat{U}(t, t_0)|x_0\rangle \langle x_0|\psi\rangle_0. \quad (6)$$

$\langle x|\psi\rangle_t$ represents the usual Schrödinger wave function

$$\langle x|\psi\rangle_t = \psi(x, t). \quad (7)$$

The fundamental quantity which marks the starting point of Feynman's reformulation of quantum mechanics is the propagator

$$G(x, t; x_0, t_0) = \langle x|\hat{U}(t, t_0)|x_0\rangle. \quad (8)$$

By the aid of (6) this propagator G can be interpreted as the wave function of a particle at time t which was strictly localized at time t_0 at the position x_0 .

If we want to guarantee causality in the time t we have to use the causal propagator G_c instead of the propagator G :

$$G_c(x, t; x_0, t_0) = \theta(t-t_0) \cdot G(x, t; x_0, t_0), \quad (9)$$

where we used the Heaviside-function θ :

$$\theta(t-t_0) = \begin{cases} 1; & t > t_0 \\ 0; & t \leq t_0 \end{cases}. \quad (10)$$

The Eqs. (4), (8) and (9) yield then a direct relation between the Hamiltonian $\hat{H}$ and the causal propagator G_c :

$$G_c(x, t; x_0, t_0) = \theta(t-t_0) \cdot \langle x|\exp \left\{ -\frac{i}{\hbar} \hat{H}(t-t_0) \right\}|x_0\rangle. \quad (11)$$

Equation (11) can be used to derive a path integral for G_c . Its continuous version in the Hamiltonian representation reads:

$$G_c(x, t; x_0, t_0) = \theta(t-t_0) \cdot \int_{x(t_0)=x_0}^{x(t)=x} D'x[\tau] \int D\left(\frac{p}{2\pi\hbar}\right)[\tau] \exp \left\{ \frac{i}{\hbar} S \right\}, \quad (12)$$

and S assumes the form of the classical action

$$S = \int_{t_0}^t d\tau \{ \dot{x}(\tau) p(\tau) - H(p(\tau), x(\tau)) \}, \quad (13)$$

where H is the classical Hamiltonian corresponding to the Hamiltonian $\hat{H}$ in Eq. (2).

The precise content of the Hamiltonian representation of the path integral can be understood from a limiting procedure which is based on an appropriate slicing

of the time-axis. If the time interval $[t_0, t]$ is divided into N pieces $[t_n, t_{n+1}]$ of equal length ε where n runs from 0 to $N-1$,

$$\varepsilon = \frac{t-t_0}{N} \quad \text{with } t_{n=0}=t_0 \quad \text{and} \quad t_{n=N}=t, \quad (14)$$

the Eqs. (12) and (13) can be written in a discrete version:

$$G_c(x, t; x_0, t_0) = \theta(t-t_0) \cdot \lim_{\varepsilon} \left\{ \prod_{n=1}^{N-1} \int dx_n \right\} \cdot \left\{ \prod_{n=0}^{N-1} \int \frac{dp_n}{2\pi\hbar} \right\} \exp \left\{ \frac{i}{\hbar} S^{(N)} \right\}, \quad (15)$$

$$S^{(N)} = \varepsilon \sum_{n=0}^{N-1} \left\{ p_n \frac{x_{n+1} - x_n}{\varepsilon} - H(p_n, x_n) \right\}. \quad (16)$$

In (15) we introduced for the limiting procedure the abbreviation

$$\lim_{\varepsilon} \bullet = \lim_{\substack{\varepsilon \rightarrow 0, \\ N \rightarrow \infty, \\ \varepsilon N = t-t_0}} \bullet. \quad (17)$$

The asymmetry in the integration over momenta and coordinates occurring in the discrete version (15) is reflected in the continuous version (12) by the prime in the functional integration over coordinates.

Equations (15) and (16) can also be interpreted in the different way that the long-time propagator $G_c(x, t; x_0, t_0)$ is built up from short-time propagators $G(x_{n+1}, t_{n+1}; x_n, t_n)$:

$$G_c(x, t; x_0, t_0) = \theta(t-t_0) \cdot \lim_{\varepsilon} \left\{ \prod_{n=1}^{N-1} \int dx_n \right\} \cdot \left\{ \prod_{n=0}^{N-1} G(x_{n+1}, t_{n+1}; x_n, t_n) \right\}. \quad (18)$$

The short-time propagators can be read off from (15) and (16):

$$G(x_{n+1}, t_{n+1}; x_n, t_n) = \int \frac{dp_n}{2\pi\hbar} \exp \left\{ \frac{i}{\hbar} [p_n(x_{n+1} - x_n) - \varepsilon H(p_n, x_n)] \right\}. \quad (19)$$

In the following we shall also make use of the Lagrangian representation of the path integral. Its short-time propagator is constructed from the Hamiltonian representation (19) by integrating out momenta p_n and using the explicit form (2) of the Hamiltonian $\hat{H}$:

$$G(x_{n+1}, t_{n+1}; x_n, t_n) = \sqrt{\frac{m}{2\pi i \hbar \varepsilon}} \cdot \exp \left\{ \frac{i}{\hbar} \varepsilon \left[\frac{m}{2} \left(\frac{x_{n+1} - x_n}{\varepsilon} \right)^2 - V(x_n) \right] \right\}. \quad (20)$$

The Lagrangian representation of the causal long-time propagator G_c is obtained by inserting the short-time propagator (20) in Eq. (18). Here we mention the continuous representation of this path integral:

$$G_c(x, t; x_0, t_0) = \theta(t-t_0) \cdot \int_{x(t_0)=x_0}^{x(t)=x} Dx[\tau] \exp \left\{ \frac{i}{\hbar} S \right\}, \quad (21)$$

$$S = \int_{t_0}^t d\tau \left\{ \frac{m}{2} \dot{x}^2(\tau) - V(x(\tau)) \right\}. \quad (22)$$

The introduction of short-time propagators offers great advantages in the treatment of path integrals. The reason is that because of the limiting procedure (17) short-time propagators need only be known up to order ε . Furthermore the partial differential equation related to the long-time propagator can be immediately established from the knowledge of the short-time propagator up to this order.

3. Transformation of time

In quantum mechanics time is usually considered as a given physical quantity. Here, however, we regard the physical time t as an additional degree of freedom which is at our disposal similar to the degree of freedom of the space. This can be achieved by a reparametrisation of the extended space-time system with the so-called pseudo-time s . The elaboration of this idea finally results in an interesting mapping of the physical time t to the pseudo-time s .

3.1. Extension of the Hilbert space

In quantum mechanics time t serves purely as a parameter to describe the succession of the state vectors $|\psi\rangle_t$ in the underlying Hilbert space $\mathcal{H}_1$ which is spanned for instance by the eigenvectors $|x\rangle$ of the position operator $\hat{x}$. In an extension of that concept we now consider the time t as an additional coordinate similar to the position x . This can be realized by formally introducing another Hilbert space $\mathcal{H}_2$ where we choose the eigenvectors $|t\rangle$ of the time operator $\hat{t}$ defined by

$$\hat{t}|t\rangle = t|t\rangle \quad (23)$$

as a complete and orthonormal basis.

To explicitly express the similarity of the position x and the time t in our notation, we regard the extended Hilbert space $\mathcal{H}$ which is constructed as the direct sum of the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. A basis for $\mathcal{H}$ is then given by

$$|x, t\rangle = |x\rangle \cdot |t\rangle. \quad (24)$$

If the states of the system are described by the state vectors $|\Psi\rangle_s$ in the extended Hilbert space $\mathcal{H}$, it becomes possible to introduce a new quantity s , the so-called pseudo-time, for their parametrization. Then the space-

time-representation of the state vector $|\Psi\rangle_s$, corresponding to the basis (24), reads:

$$\langle x, t | \Psi \rangle_s = \Psi(x, t, s). \quad (25)$$

In the following subsection we want to reflect upon the realization and the consequences for the causal propagator $G_c(x, t; x_0, t_0)$ due to this extension of the underlying function space.

3.2. Consequences for the causal propagator

As we shall see it becomes possible to derive the causal propagator G_c , related to $\mathcal{H}_1$, from a more general causal propagator $G_c^{(1)}$ which corresponds to the extended Hilbert space $\mathcal{H}$. This can be achieved by the successive application of simple identities. In a first step we apply a delta-function according to

$$G_c(x, t; x_0, t_0) = \int_0^{+\infty} ds G_c(x, t; x_0, t_0) \cdot \delta(t - t_0 - s). \quad (26)$$

To profit from this identity we note that the eigenvectors $|E\rangle$ of the energy operator $\hat{E}$, defined by

$$\hat{E}|E\rangle = E|E\rangle, \quad (27)$$

have the time representation

$$\langle t | E \rangle = \frac{1}{\sqrt{2\pi\hbar}} \cdot \exp\left\{-\frac{i}{\hbar} E t\right\}. \quad (28)$$

Under these circumstances the delta-function in (26) takes the form

$$\delta(t - t_0 - s) = \langle t | \exp\left\{+\frac{i}{\hbar} \hat{E} s\right\} | t_0 \rangle. \quad (29)$$

Because of the time independence of the Hamiltonian $\hat{H}$, we can use (11) and (29) to write (26) in the form

$$G_c(x, t; x_0, t_0) = \int_{-\infty}^{+\infty} ds \theta(s) \cdot \langle x, t | \exp\left\{-\frac{i}{\hbar} (\hat{H} - \hat{E}) s\right\} | x_0, t_0 \rangle. \quad (30)$$

Equation (30) can formally be integrated:

$$G_c(x, t; x_0, t_0) = \langle x, t | \hat{R} | x_0, t_0 \rangle, \quad (31)$$

$$\hat{R} = \frac{-i\hbar}{\hat{H} - \hat{E}}. \quad (32)$$

The operator $\hat{R}$ has to be understood in the sense

$$\hat{R} = \lim_{\eta \downarrow 0} \frac{-i\hbar}{\hat{H} - \hat{E} - i\eta}, \quad (33)$$

where the introduced positive and real quantity η expresses causality in the pseudo-time domain.

3.3. General perspective from the extension

We suppose that $\hat{f}_r$ and $\hat{f}_l$ are invertible but otherwise arbitrary operators. Then the operator identity

$$(\hat{A}\hat{B})^{-1} = \hat{B}^{-1} \hat{A}^{-1} \quad (34)$$

allows us to rewrite (32):

$$\hat{R} = \hat{f}_r \cdot \frac{-i\hbar}{\hat{f}_l(\hat{H} - \hat{E})\hat{f}_r} \cdot \hat{f}_l. \quad (35)$$

If we consider $\hat{f}_r$ and $\hat{f}_l$ as functions of the position operator $\hat{x}$ and the time operator $\hat{t}$,

$$\hat{f}_r = f_r(\hat{x}, \hat{t}) \quad \text{and} \quad \hat{f}_l = f_l(\hat{x}, \hat{t}), \quad (36)$$

the Eqs. (31) and (35) can equally be written as

$$G_c(x, t; x_0, t_0) = f_r(x, t) f_l(x_0, t_0) \cdot \int_0^{+\infty} ds G_c^{(1)}(x, t, s; x_0, t_0, 0), \quad (37)$$

$$G_c^{(1)}(x, t, s; x_0, t_0, 0) = \theta(s) \cdot \langle x, t | \exp\left\{-\frac{i}{\hbar} \hat{f}_l(\hat{H} - \hat{E}) \hat{f}_r s\right\} | x_0, t_0 \rangle. \quad (38)$$

Equation (37) represents a relationship between the causal propagator $G_c(x, t; x_0, t_0)$ in the original Hilbert space $\mathcal{H}_1$ and a new causal propagator $G_c^{(1)}(x, t, s; x_0, t_0, 0)$ in the extended Hilbert space $\mathcal{H}$. Furthermore, a generalization of the considerations in Sect. 2 leads directly to a proper interpretation of (38). The causal propagator $G_c^{(1)}(x, t, s; x_0, t_0, 0)$ corresponds to the initial value problem

$$i\hbar \partial_s |\Psi^{(1)}\rangle_s = \hat{f}_l(\hat{H} - \hat{E}) \hat{f}_r |\Psi^{(1)}\rangle_s \quad \text{with} \quad |\Psi^{(1)}\rangle_{s=0} = |\Psi^{(1)}\rangle_0 \quad (39)$$

in $\mathcal{H}$ and maps the space-time representation of the initial state $|\Psi^{(1)}\rangle_0$ to the space-time representation of the solution $|\Psi^{(1)}\rangle_s$:

$$\langle x, t | \Psi^{(1)} \rangle_s = \int dx_0 \int dt_0 G_c^{(1)}(x, t, s; x_0, t_0, 0) \cdot \langle x_0, t_0 | \Psi^{(1)} \rangle_0. \quad (40)$$

The two operators $\hat{f}_l$ and $\hat{f}_r$ in (39) offer new perspectives for calculations concerning the evaluation of path integrals, as will be seen later on.

3.4. Discretized path integral representation of the causal propagator

Here we shall derive a path integral representation for the causal propagator $G_c^{(1)}(x, t, s; x_0, t_0, 0)$. If we slice the s -axis in analogy to (14) into N pieces of equal length ε_s , the causal long-time propagator $G_c^{(1)}(x, t, s; x_0, t_0, 0)$ can be built up from short-time propagators of type $G^{(1)}(x_{n+1}, t_{n+1}, \varepsilon_s; x_n, t_n, 0)$:

$$G_c^{(1)}(x, t, s; x_0, t_0, 0) = \theta(s) \cdot \lim_{\varepsilon_s} \left\{ \prod_{n=1}^{N-1} \int dx_n \int dt_n \right\} \cdot \left\{ \prod_{n=0}^{N-1} G^{(1)}(x_{n+1}, t_{n+1}, \varepsilon_s; x_n, t_n, 0) \right\}. \quad (41)$$

In accordance with (38) the short-time propagator is then given by

$$G^{(1)}(x_{n+1}, t_{n+1}, \varepsilon_s; x_n, t_n, 0) = \langle x_{n+1}, t_{n+1} | \exp \left\{ -\frac{i}{\hbar} \hat{f}_l(\hat{H} - \hat{E}) \hat{f}_r \cdot \varepsilon_s \right\} | x_n, t_n \rangle. \quad (42)$$

A further evaluation of the short-time propagator $G^{(1)}(x_{n+1}, t_{n+1}, \varepsilon_s; x_n, t_n, 0)$ becomes possible for the following reason. Because of the limiting procedure in (41) all short-time propagators which are equal up to the first order in ε_s lead to the same causal long-time propagator $G_c^{(1)}(x, t, s; x_0, t_0, 0)$. Using the Hamiltonian $\hat{H}$ in (2) allows us to expand the exponential in (42) up to order ε_s , to partly evaluate the matrix elements and to rearrange the result again as a product of exponentials. We obtain:

$$\begin{aligned} G^{(1)}(x_{n+1}, t_{n+1}, \varepsilon_s; x_n, t_n, 0) &= \langle x_{n+1}, t_{n+1} | \\ &\cdot \exp \left\{ -\frac{i}{\hbar} f_l(x_{n+1}, t_{n+1}) \cdot \frac{\hat{p}^2}{2m} \cdot f_r(x_n, t_n) \varepsilon_s \right\} \\ &\cdot \exp \left\{ -\frac{i}{\hbar} f_l(x_{n+1}, t_{n+1}) V(x_n) f_r(x_n, t_n) \varepsilon_s \right\} \\ &\cdot \exp \left\{ +\frac{i}{\hbar} f_l(x_{n+1}, t_{n+1}) \hat{E} f_r(x_n, t_n) \varepsilon_s \right\} | x_n, t_n \rangle. \end{aligned} \quad (43)$$

The next step consists in the replacement of the remaining operators by their eigenvalues. To this end it is necessary to introduce another complete basis of the extended Hilbert space built by

$$|p, E\rangle = |p\rangle \cdot |E\rangle, \quad (44)$$

where $|p\rangle$ and $|E\rangle$ are the eigenvectors of the momentum and the energy operator, respectively. A two-fold application of the completeness relation represented by

$$\int dp \int dE |p, E\rangle \langle p, E| = 1 \quad (45)$$

and the space-time representation of the basis vectors $|p, E\rangle$

$$\langle x, t | p, E \rangle = \frac{1}{2\pi\hbar} \cdot \exp \left\{ +\frac{i}{\hbar} (px - Et) \right\} \quad (46)$$

are needed. Equations (45) and (46) together with a complete integration over the energy eigenstates then transform the short-time propagator into the form:

$$\begin{aligned} G^{(1)}(x_{n+1}, t_{n+1}, \varepsilon_s; x_n, t_n, 0) &= \delta(t_{n+1} - t_n - \varepsilon_s f_l(x_{n+1}, t_{n+1}) f_r(x_n, t_n)) \\ &\cdot \int \frac{dp_n}{2\pi\hbar} \exp \left\{ \frac{i}{\hbar} [p_n(x_{n+1} - x_n) \right. \\ &\quad \left. - \varepsilon_s f_l(x_{n+1}, t_{n+1}) H(p_n, x_n) f_r(x_n, t_n)] \right\}, \end{aligned} \quad (47)$$

As a consequence the long-time propagator (41) can be written as

$$\begin{aligned} G_c^{(1)}(x, t, s; x_0, t_0, 0) &= \theta(s) \cdot \lim_{\varepsilon_s} \left\{ \prod_{n=1}^{N-1} \int dx_n \int dt_n \right\} \left\{ \prod_{n=0}^{N-1} \int \frac{dp_n}{2\pi\hbar} \right\} \\ &\cdot \exp \left\{ \frac{i}{\hbar} S^{(N)} \right\} \cdot \left\{ \prod_{n=0}^{N-1} \delta(t_{n+1} - t_n - \varepsilon_s \right. \\ &\quad \left. \cdot f_l(x_{n+1}, t_{n+1}) f_r(x_n, t_n) \right\}, \end{aligned} \quad (48)$$

where

$$\begin{aligned} S^{(N)} &= \varepsilon_s \sum_{n=0}^{N-1} \left\{ p_n \frac{x_{n+1} - x_n}{\varepsilon_s} \right. \\ &\quad \left. - f_l(x_{n+1}, t_{n+1}) \cdot H(p_n, x_n) f_r(x_n, t_n) \right\}. \end{aligned} \quad (49)$$

Equations (48) and (49) lead to an important result. The N delta-functions in time secure the relation

$$\varepsilon = \varepsilon_s \cdot f_l(x_{n+1}, t_{n+1}) \cdot f_r(x_n, t_n) \quad \text{for } n=0, 1, \dots, N-1 \quad (50)$$

between the pseudo-time element ε_s and the element of the physical time ε . In the continuous version, where the number of intervals goes to infinity, this difference equation leads to the corresponding differential equation

$$\frac{d}{ds} t(s) = f_l(x(s), t(s)) \cdot f_r(x(s), t(s)). \quad (51)$$

Equation (51) defines a local time transformation from t to s [11]. This formula can be interpreted in the following way. Whereas in the physical time domain t each space point is equipped with the same time scale, the time scale with respect to the pseudo-time s becomes space dependent.

3.5. Further specialization

It is our purpose to simplify the calculations by assuming

$$f_l(x, t) = f(x, t) \quad \text{and} \quad f_r(x, t) = 1. \quad (52)$$

The insertion of (52) in the previous formulae gives us the opportunity to summarize our temporary results. First, the extension of the original Hilbert space $\mathcal{H}_1$ to the Hilbert space $\mathcal{H}$ allowed us to express the causal propagator G_c in terms of a new causal propagator $G_c^{(1)}$.

From (37) and (52) we get:

$$G_c(x, t; x_0, t_0) = f(x_0, t_0) \cdot \int_0^{+\infty} ds G_c^{(1)}(x, t, s; x_0, t_0, 0). \quad (53)$$

The slicing of the s -axis then yielded a path integral for $G_c^{(1)}(x, t, s; x_0, t_0, 0)$. With (48), (49) and (52) we obtain – after an integration over the momenta – the Lagrangian representation:

$$\begin{aligned} G_c^{(1)}(x, t, s; x_0, t_0, 0) \\ = \theta(s) \cdot \lim_{\varepsilon_s} \left\{ \prod_{n=1}^{N-1} \int dx_n \int dt_n \right\} \\ \cdot \left\{ \prod_{n=0}^{N-1} \sqrt{\frac{m}{2\pi i \hbar \varepsilon_s f(x_{n+1}, t_{n+1})}} \right. \\ \cdot \delta(t_{n+1} - t_n - \varepsilon_s f(x_{n+1}, t_{n+1})) \Big\} \\ \cdot \exp \left\{ \frac{i}{\hbar} \sum_{n=0}^{N-1} f(x_{n+1}, t_{n+1}) \varepsilon_s \right. \\ \cdot \left. \left[\frac{m}{2} \left(\frac{x_{n+1} - x_n}{f(x_{n+1}, t_{n+1}) \varepsilon_s} \right)^2 - V(x_n) \right] \right\}. \quad (54) \end{aligned}$$

4. Transformation of coordinate

This part is devoted to the exploration of an additional transformation involving space. Our aim is to implement an occasionally time-dependent mapping from the original coordinate x to the new coordinate q . In the Lagrangian form of the path integral this is achieved by an explicitly time-dependent point transformation in the configuration space

$$x = h(q, t), \quad (55)$$

where h denotes an arbitrary invertible function.

4.1. Consequences for the causal propagator

We insert (55) in the expression (54) for the causal propagator $G_c^{(1)}$:

$$\begin{aligned} G_c^{(1)}(x, t, s; x_0, t_0, 0) \\ = \frac{\theta(s)}{h'(h^{-1}(x_0, t_0), t_0)} \cdot \lim_{\varepsilon_s} \left\{ \prod_{n=1}^{N-1} \int dq_n \int dt_n \right\} \\ \cdot \left\{ \prod_{n=0}^{N-1} \sqrt{\frac{m}{2\pi i \hbar \varepsilon_s}} \cdot \frac{h'(q_{n+1}, t_{n+1})}{\sqrt{f(h(q_{n+1}, t_{n+1}), t_{n+1})}} \right. \\ \cdot \delta(t_{n+1} - t_n - f(h(q_{n+1}, t_{n+1}), t_{n+1}) \varepsilon_s) \cdot \frac{h'(q_n, t_n)}{h'(q_{n+1}, t_{n+1})} \Big\} \\ \cdot \exp \left\{ \frac{i}{\hbar} \sum_{n=0}^{N-1} f(h(q_{n+1}, t_{n+1}), t_{n+1}) \varepsilon_s \right. \\ \cdot \left. \left[\frac{m}{2} \left(\frac{h(q_{n+1}, t_{n+1}) - h(q_n, t_n)}{f(h(q_{n+1}, t_{n+1}), t_{n+1}) \varepsilon_s} \right)^2 - V(h(q_n, t_n)) \right] \right\}. \quad (56) \end{aligned}$$

Here h' stands for the partial derivative of the function h with respect to the first variable and h^{-1} denotes the inverse of the point transformation (55) according to

$$q = h^{-1}(x, t). \quad (57)$$

In (56) the reciprocal of $h'(h^{-1}(x_0, t_0), t_0)$ appears in front of the limiting procedure. This arrangement of factors motivates one to combine the point transformation (55) with a corresponding mapping of the previous causal propagator $G_c^{(1)}(x, t, s; x_0, t_0, 0)$ to a new causal propagator which we call $G^{(2)}(q, t, s; q_0, t_0, 0)$:

$$\begin{aligned} G_c^{(1)}(x, t, s; x_0, t_0, 0) \\ = \frac{1}{h'(h^{-1}(x_0, t_0), t_0)} \cdot G_c^{(2)}(h^{-1}(x, t), t, s; h^{-1}(x_0, t_0), t_0, 0). \quad (58) \end{aligned}$$

From (56) and (58) it is obvious that the new causal propagator $G_c^{(2)}$ can be built up again from short-time propagators $G_c^{(2)}$:

$$\begin{aligned} G_c^{(2)}(q, t, s; q_0, t_0, 0) \\ = \theta(s) \cdot \lim_{\varepsilon_s} \left\{ \prod_{n=1}^{N-1} \int dq_n \int dt_n \right\} \\ \cdot \left\{ \prod_{n=0}^{N-1} G^{(2)}(q_{n+1}, t_{n+1}, \varepsilon_s; q_n, t_n, 0) \right\}. \quad (59) \end{aligned}$$

The new short-time propagator $G^{(2)}$ follows from (56) in the form

$$\begin{aligned} G^{(2)}(q_{n+1}, t_{n+1}, \varepsilon_s; q_n, t_n, 0) \\ = \sqrt{\frac{m}{2\pi i \hbar \varepsilon_s}} \cdot \delta(t_{n+1} - t_n - f(h(q_{n+1}, t_{n+1}), t_{n+1}) \varepsilon_s) \\ \cdot \frac{h'(q_n, t_n)}{h'(q_{n+1}, t_{n+1})} \cdot \frac{h'(q_{n+1}, t_{n+1})}{\sqrt{f(h(q_{n+1}, t_{n+1}), t_{n+1})}} \\ \cdot \exp \left\{ \frac{i}{\hbar} f(h(q_{n+1}, t_{n+1}), t_{n+1}) \varepsilon_s \right. \\ \cdot \left. \left[\frac{m}{2} \left(\frac{h(q_{n+1}, t_{n+1}) - h(q_n, t_n)}{f(h(q_{n+1}, t_{n+1}), t_{n+1}) \varepsilon_s} \right)^2 - V(h(q_n, t_n)) \right] \right\}. \quad (60) \end{aligned}$$

4.2. Expansion of the short-time propagator

In a series of steps we intend to derive a simpler expression for the short-time propagator (60). For this purpose we recall the fact that each short-time propagator is related to a partial differential equation and vice versa. To prepare for the technique which allows us to switch from the short-time propagator $G^{(2)}$ to its corresponding partial differential equation we expand $G^{(2)}$ into powers of ε_s up to the first order. In doing this we have to take into account that the differences in the new coordinate $q_{n+1} - q_n$ and the time $t_{n+1} - t_n$ also contribute to this expansion according to the following rules:

1. From the delta-function in (60) we conclude that

$$\Delta t_n = t_{n+1} - t_n \sim \varepsilon_s. \quad (61)$$

2. As we shall later show in detail by the evaluation of the Fresnel integrals (81) we have

$$\Delta q_n = q_{n+1} - q_n \sim \varepsilon_s^{\frac{3}{2}}. \quad (62)$$

It turns out that the expansion into powers of ε_s becomes a lengthy but straight-forward calculation. Therefore we shall only present the final result. To guarantee a concise presentation we choose additional abbreviations for the partial derivatives of the function h with respect to each variable:

$$h' = \frac{\partial}{\partial q_{n+1}} h(q_{n+1}, t_{n+1}) \quad \text{and} \quad \dot{h} = \frac{\partial}{\partial t_{n+1}} h(q_{n+1}, t_{n+1}). \quad (63)$$

The expression then reads:

$$\begin{aligned} G^{(2)}(q_{n+1}, t_{n+1}, \varepsilon_s; q_n, t_n, 0) \\ = \sqrt{\frac{m}{2\pi i \hbar \varepsilon_s}} \cdot \delta(\Delta t_n - f(h, t_{n+1}) \varepsilon_s) \cdot \frac{h'}{\sqrt{f(h, t_{n+1})}} \\ \cdot \left\{ 1 - \frac{h''}{h'} \cdot \Delta q_n + \frac{1}{2} \frac{h'''}{h'} \cdot \Delta q_n^2 - \frac{\dot{h}'}{h'} \cdot \Delta t_n \right\} \\ \cdot \exp \left\{ -\frac{m}{2i\hbar \varepsilon_s} \cdot \frac{h'^2}{f(h, t_{n+1})} \right. \\ \cdot \left[\Delta q_n^2 - \frac{h''}{h'} \cdot \Delta q_n^3 + \left(\frac{1}{3} \frac{h'''}{h'} + \frac{1}{4} \frac{h''^2}{h'^2} \right) \right. \\ \cdot \Delta q_n^4 + 2 \frac{\dot{h}}{h'} \cdot \Delta q_n \Delta t_n - 2 \frac{\dot{h}'}{h'} \cdot \Delta q_n^2 \Delta t_n \\ \left. \left. - \frac{\dot{h} h''}{h'^2} \cdot \Delta q_n^2 \Delta t_n + \frac{\dot{h}^2}{h'^2} \cdot \Delta t_n^2 \right] - \frac{i}{\hbar} f(h, t_{n+1}) \varepsilon_s V(h) \right\}. \quad (64) \end{aligned}$$

4.3. The first condition

Until now the functions f and h have been considered as invertible but otherwise arbitrary functions of x , t and q , t . We observe, however, that (64) can be substantially simplified in two different respects by imposing what we call the first condition:

$$f(h(q, t), t) = h'(q, t)^2. \quad (65)$$

Applying this relation between the coordinate and the time transformation in the expression (64) for the short-time propagator and substituting $\Delta t_n = \varepsilon_s h'^2$ according to the prescription of the delta-function we find

$$\begin{aligned} G^{(2)}(q_{n+1}, t_{n+1}, \varepsilon_s; q_n, t_n, 0) \\ = \sqrt{\frac{m}{2\pi i \hbar \varepsilon_s}} \cdot \delta(\Delta t_n - \varepsilon_s h'^2) \\ \cdot \left\{ 1 - \frac{h''}{h'} \cdot \Delta q_n + \frac{1}{2} \frac{h'''}{h'} \cdot \Delta q_n^2 - \dot{h}' h' \varepsilon_s \right\} \end{aligned}$$

$$\begin{aligned} \cdot \exp \left\{ -\frac{m}{2i\hbar \varepsilon_s} \cdot \left[\Delta q_n^2 - \frac{h''}{h'} \cdot \Delta q_n^3 \right. \right. \\ \left. \left. + \left(\frac{1}{3} \frac{h'''}{h'} + \frac{1}{4} \frac{h''^2}{h'^2} \right) \cdot \Delta q_n^4 + 2 \dot{h} h' \cdot \Delta q_n \varepsilon_s - 2 \dot{h}' h' \cdot \Delta q_n^2 \varepsilon_s \right. \right. \\ \left. \left. - \dot{h} h'' \cdot \Delta q_n^2 \varepsilon_s + \dot{h}^2 h'^2 \varepsilon_s^2 \right] - \frac{i}{\hbar} \varepsilon_s h'^2 V(h) \right\}. \quad (66) \end{aligned}$$

Indeed the essence of the first condition (65) consists in the fact that we recover in the first term of the exponential in (66) the expression of the free particle in the new coordinate q . We consider this as a first but essential step to recover the Lagrangian path integral of a usual quantum mechanical system. Furthermore the term in the above expression corresponding to the free particle allows us in the following to result in the Fresnel integrals (81). As we shall see, this validates the expansion rule (62).

The causal long-time propagator $G_c^{(2)}$ can now be derived from Eqs. (59) and (66):

$$\begin{aligned} G_c^{(2)}(q, t, s; q_0, t_0, 0) \\ = \theta(s) \cdot \lim_{\varepsilon_s} \left\{ \prod_{n=1}^{N-1} \int dq_n \int dt_n \right\} \\ \cdot \left\{ \prod_{n=0}^{N-1} \sqrt{\frac{m}{2\pi i \hbar \varepsilon_s}} \cdot \delta(\Delta t_n - \varepsilon_s h'^2) \right. \\ \cdot \left\{ 1 - \frac{h''}{h'} \cdot \Delta q_n + \frac{1}{2} \frac{h'''}{h'} \cdot \Delta q_n^2 - \dot{h}' h' \varepsilon_s \right\} \\ \cdot \exp \left\{ -\frac{m}{2i\hbar \varepsilon_s} \cdot \left[\Delta q_n^2 - \frac{h''}{h'} \cdot \Delta q_n^3 + \left(\frac{1}{3} \frac{h'''}{h'} + \frac{1}{4} \frac{h''^2}{h'^2} \right) \right. \right. \\ \cdot \Delta q_n^4 + 2 \dot{h} h' \cdot \Delta q_n \varepsilon_s - 2 \dot{h}' h' \cdot \Delta q_n^2 \varepsilon_s \\ \left. \left. - \dot{h} h'' \cdot \Delta q_n^2 \varepsilon_s + \dot{h}^2 h'^2 \varepsilon_s^2 \right] - \frac{i}{\hbar} \varepsilon_s h'^2 V(h) \right\} \right\}. \quad (67) \end{aligned}$$

$G_c^{(2)}$ is the propagator of the wave function

$$\langle q, t | \Psi^{(2)} \rangle_s = \Psi^{(2)}(q, t, s) \quad (68)$$

and transforms in complete analogy to (40) between states with successive values of the pseudo-time s :

$$\begin{aligned} \Psi^{(2)}(q, t, s) \\ = \int dq_0 \int dt_0 G_c^{(2)}(q, t, s; q_0, t_0, 0) \cdot \Psi^{(2)}(q_0, t_0, 0). \quad (69) \end{aligned}$$

5. Transformation of the wave function

If we were to calculate the partial differential equation corresponding to the short-time propagator (66), then we would be encountered with two problems:

1. Because of the transformation of time the new Hamiltonian need no longer be hermitian.
2. The new Hamiltonian may contain a first order partial derivative with respect to the new coordinate q .

In order to remove at least one of these two problems we introduce a new and arbitrary complex function $g = g(q, t)$.

5.1. Introduction of a new function

We consider a linear transformation of the wave function $\Psi^{(2)}$ of (68) to a new one denoted by $\Psi^{(3)}$:

$$\Psi^{(2)}(q, t, s) = g(q, t) \cdot \Psi^{(3)}(q, t, s). \quad (70)$$

If we insert (70) into the time evolution (69) of $\Psi^{(2)}$, we get the time evolution of $\Psi^{(3)}$:

$$\begin{aligned} & \Psi^{(3)}(q, t, s) \\ &= \int dq_0 \int dt_0 \left\{ \frac{g(q_0, t_0)}{g(q, t)} \cdot G_c^{(2)}(q, t, s; q_0, t_0, 0) \right\} \\ & \cdot \Psi^{(3)}(q_0, t_0, 0). \end{aligned} \quad (71)$$

The linear transformation of the wave function in (70) obviously has consequences for the propagators. The causal propagator $G_c^{(2)}$ is mapped to a new one which we denote by $G_c^{(3)}$:

$$G_c^{(3)}(q, t, s; q_0, t_0, 0) = \frac{g(q_0, t_0)}{g(q, t)} \cdot G_c^{(2)}(q, t, s; q_0, t_0, 0). \quad (72)$$

5.2. Evaluation of the short-time propagator

Using the obvious identity

$$1 = \frac{g(q, t)}{g(q_0, t_0)} \cdot \left\{ \prod_{n=0}^{N-1} \frac{g(q_n, t_n)}{g(q_{n+1}, t_{n+1})} \right\} \quad (73)$$

in (67) and comparing the result with the relationship (72) of the causal long-time propagators $G_c^{(2)}$ and $G_c^{(3)}$, we deduce a similar relationship for the corresponding short-time propagators $G^{(2)}$ and $G^{(3)}$:

$$\begin{aligned} & G^{(3)}(q_{n+1}, t_{n+1}, \varepsilon_s; q_n, t_n, 0) \\ &= \frac{g(q_n, t_n)}{g(q_{n+1}, t_{n+1})} \cdot G^{(2)}(q_{n+1}, t_{n+1}, \varepsilon_s; q_n, t_n, 0). \end{aligned} \quad (74)$$

In accordance with previous considerations in Subsect. 4.3 we are interested in the short-time propagator $G^{(3)}$ to an accuracy of order ε_s . We only have to deal with the quotient $g(q_n, t_n)/g(q_{n+1}, t_{n+1})$, because $G^{(2)}$ was treated in (66). Applying again the two rules (61) and (62) we get with an abbreviation analogous to (63):

$$\frac{g(q_n, t_n)}{g(q_{n+1}, t_{n+1})} = 1 - \frac{g'}{g} \cdot \Delta q_n + \frac{1}{2} \frac{g''}{g} \cdot \Delta q_n^2 - \frac{\dot{g}}{g} \cdot h'^2 \varepsilon_s. \quad (75)$$

A combination of (66), (74) and (75) leads directly to the short-time propagator $G^{(3)}$:

$$\begin{aligned} & G^{(3)}(q_{n+1}, t_{n+1}, \varepsilon_s; q_n, t_n, 0) \\ &= \sqrt{\frac{m}{2\pi i \hbar \varepsilon_s}} \cdot \delta(\Delta t_n - \varepsilon_s h'^2) \\ & \cdot \left\{ 1 - \frac{g'}{g} \cdot \Delta q_n + \frac{1}{2} \frac{g''}{g} \cdot \Delta q_n^2 - \frac{\dot{g}}{g} \cdot h'^2 \varepsilon_s \right\} \end{aligned}$$

$$\begin{aligned} & \cdot \left\{ 1 - \frac{h''}{h'} \cdot \Delta q_n + \frac{1}{2} \frac{h'''}{h'} \cdot \Delta q_n^2 - \dot{h}' h' \varepsilon_s \right\} \\ & \cdot \exp \left\{ -\frac{m}{2i\hbar \varepsilon_s} \left[\Delta q_n^2 - \frac{h''}{h'} \cdot \Delta q_n^3 + \left(\frac{1}{3} \frac{h'''}{h'} + \frac{1}{4} \frac{h''^2}{h'^2} \right) \right. \right. \\ & \cdot \Delta q_n^4 + 2\dot{h}' h' \cdot \Delta q_n \varepsilon_s - 2\dot{h}' h' \cdot \Delta q_n^2 \varepsilon_s \\ & \left. \left. - \dot{h}' h'' \cdot \Delta q_n^2 \varepsilon_s + \dot{h}^2 h'^2 \varepsilon_s^2 \right] - \frac{i}{\hbar} \varepsilon_s h'^2 V(h) \right\}. \end{aligned} \quad (76)$$

5.3. Determination of the partial differential equation

Now we deduce the partial differential equation for the new wave function $\Psi^{(3)}$ which corresponds to the short-time propagator $G^{(3)}$. We note that the partial derivative of the wave function $\Psi^{(3)}$ with respect to the pseudo-time s can be written as the limit

$$\begin{aligned} & \frac{\partial}{\partial s} \Psi^{(3)}(q_{n+1}, t_{n+1}, s) \\ &= \lim_{\varepsilon_s \rightarrow 0} \frac{\Psi^{(3)}(q_{n+1}, t_{n+1}, s + \varepsilon_s) - \Psi^{(3)}(q_{n+1}, t_{n+1}, s)}{\varepsilon_s}. \end{aligned} \quad (77)$$

To evaluate (77) we take into account that the two wave functions $\Psi^{(3)}(q_{n+1}, t_{n+1}, s)$ and $\Psi^{(3)}(q_{n+1}, t_{n+1}, s + \varepsilon_s)$ are related to each other by means of the short-time evolution

$$\begin{aligned} & \Psi^{(3)}(q_{n+1}, t_{n+1}, s + \varepsilon_s) \\ &= \int dq_n \int dt_n G^{(3)}(q_{n+1}, t_{n+1}, \varepsilon_s; q_n, t_n, 0) \cdot \Psi^{(3)}(q_n, t_n, s). \end{aligned} \quad (78)$$

We insert the short-time propagator $G^{(3)}$ from (76) in (78) and substitute the integration variables:

$$\begin{aligned} \xi_n(q_n) &= q_n - q_{n+1} = -\Delta q_n \quad \text{and} \\ \tau_n(t_n) &= t_n - t_{n+1} = -\Delta t_n. \end{aligned} \quad (79)$$

Applying the rules given in (61) and (62) we expand the complete integrand into powers of ε_s keeping the factor which describes the contribution of the free particle. We additionally perform the integration over the time difference τ_n to obtain the provisional result

$$\begin{aligned} & \Psi^{(3)}(q_{n+1}, t_{n+1}, s + \varepsilon_s) \\ &= \int d\xi \sqrt{\frac{m}{2\pi i \hbar \varepsilon_s}} \cdot \exp \left\{ -\frac{m}{2i\hbar \varepsilon_s} \xi^2 \right\} \\ & \cdot \left[1 - \frac{m}{2i\hbar \varepsilon_s} \cdot \left\{ \frac{h''}{h'} \xi^3 + \left(\frac{1}{3} \frac{h'''}{h'} + \frac{5}{4} \frac{h''^2}{h'^2} \right) \xi^4 \right. \right. \\ & \left. \left. - 2\dot{h}' h' \xi \varepsilon_s - 2\dot{h}' h' \xi^2 \varepsilon_s - 3\dot{h}' h'' \xi^2 \varepsilon_s + \dot{h}^2 h'^2 \varepsilon_s^2 \right\} \right. \\ & \left. + \frac{1}{2} \left(\frac{m}{2i\hbar \varepsilon_s} \right)^2 \cdot \left\{ \frac{h''^2}{h'^2} \xi^6 - 4\dot{h}' h'' \xi^4 \varepsilon_s + 4\dot{h}^2 h'^2 \xi^2 \varepsilon_s^2 \right\} \right. \\ & \left. - \frac{i}{\hbar} \varepsilon_s h'^2 V(h) + 1 + \frac{g'}{g} \xi + \left(\frac{1}{2} \frac{g''}{g} + \frac{g'}{g} \frac{h''}{h'} \right) \xi^2 \right] \end{aligned}$$

$$\begin{aligned}
& -\frac{\dot{g}}{g} h'^2 \varepsilon_s - \frac{m}{2i\hbar \varepsilon_s} \left\{ \frac{g'}{g} \frac{h''}{h'} \xi^4 - 2 \frac{g'}{g} \dot{h} h' \xi^2 \varepsilon_s \right\} \\
& + \frac{h''}{h'} \xi + \frac{1}{2} \frac{h'''}{h'} \xi^2 - \dot{h} h' \varepsilon_s \left\{ 1 + \xi \cdot \frac{\partial}{\partial q_{n+1}} + \frac{1}{2} \xi^2 \right. \\
& \cdot \frac{\partial^2}{\partial q_{n+1}^2} - h'^2 \varepsilon_s \cdot \frac{\partial}{\partial t_{n+1}} \left. \right\} \Psi^{(3)}(q_{n+1}, t_{n+1}, s). \quad (80)
\end{aligned}$$

The remaining Fresnel integrals lead to

$$\int d\xi \xi^2 \cdot \exp\{-\lambda \xi^2\} = \frac{1 \cdot 3 \cdot \dots \cdot (2n-1)}{(2\lambda)^n}$$

$$\text{with } \lambda = \frac{m}{2i\hbar \varepsilon_s}. \quad (81)$$

By the way, the integration formula (81) justifies the previously mentioned expansion rule (62).

Using (77), (80) and (81) we are able to determine the partial differential equation for the wave function $\Psi^{(3)}$:

$$\begin{aligned}
& i\hbar \frac{\partial}{\partial s} \Psi^{(3)}(q_{n+1}, t_{n+1}, s) \\
& = \left\{ -\frac{\hbar^2}{2m} \cdot \frac{\partial^2}{\partial q_{n+1}^2} - \frac{\hbar^2}{m} \left[\frac{g'}{g} - \frac{1}{2} \frac{h''}{h'} - \frac{im}{\hbar} \dot{h} h' \right] \cdot \frac{\partial}{\partial q_{n+1}} \right. \\
& \quad \left. + V^{(3)} - i\hbar h'^2 \cdot \frac{\partial}{\partial t_{n+1}} \right\} \Psi^{(3)}(q_{n+1}, t_{n+1}, s), \quad (82)
\end{aligned}$$

where we introduce the potential $V^{(3)}$ as

$$\begin{aligned}
V^{(3)} &= h'^2 V(h) - i\hbar \frac{\dot{g}}{g} h'^2 \\
& - \frac{\hbar^2}{2m} \left\{ \frac{g''}{g} - \frac{g'}{g} \left[\frac{h''}{h} + \frac{2im}{\hbar} h' \dot{h} \right] \right\}. \quad (83)
\end{aligned}$$

5.4. The second condition

Until now we have not specified the complex function g . To conclusively obtain a new formal Schrödinger equation for the wave function it is, however, indispensable that the first order partial derivative with respect to the new coordinate q_{n+1} in (82) vanishes. To simplify (82) in this spirit we impose the second condition:

$$\frac{g'}{g} = \frac{1}{2} \frac{h''}{h'} + \frac{im}{\hbar} \dot{h} h'. \quad (84)$$

Integration yields a relationship between the coordinate and wave function transformation:

$$g(q, t) = \sqrt{h'(q, t)} \cdot \exp \left\{ \frac{im}{\hbar} \int \dot{h}(\tilde{q}, t) h'(\tilde{q}, t) d\tilde{q} \right\}. \quad (85)$$

Using (85) the partial differential (82) and the potential (83) are converted into

$$\begin{aligned}
& i\hbar \frac{\partial}{\partial s} \Psi^{(3)}(q_{n+1}, t_{n+1}, s) \\
& = \left\{ -\frac{\hbar^2}{2m} \cdot \frac{\partial^2}{\partial q_{n+1}^2} + V^{(3)} - i\hbar h'^2 \cdot \frac{\partial}{\partial t_{n+1}} \right\} \\
& \cdot \Psi^{(3)}(q_{n+1}, t_{n+1}, s), \quad (86)
\end{aligned}$$

$$\begin{aligned}
V^{(3)} &= h'^2 \int h' \left\{ m \dot{h} + \frac{\partial V(h)}{\partial h} \right\} dq_{n+1} \\
& - i\hbar \dot{h} h' + \frac{\hbar^2}{m} \left\{ \frac{3}{8} \frac{h''^2}{h'^2} - \frac{1}{4} \frac{h'''}{h'} \right\}. \quad (87)
\end{aligned}$$

5.5. Propagator transformation formula

We return from the level of the partial differential equation back to the related path integrals and obtain for the causal propagator $G_c^{(3)}$:

$$\begin{aligned}
& G_c^{(3)}(q, t, s; q_0, t_0, 0) \\
& = \theta(s) \lim_{\varepsilon_s} \left\{ \prod_{n=1}^{N-1} \int dq_n \int dt_n \right\} \\
& \cdot \left\{ \prod_{n=0}^{N-1} \delta(t_{n+1} - t_n - \varepsilon_s h'(q_{n+1}, t_{n+1})^2) \sqrt{\frac{m}{2\pi i \hbar \varepsilon_s}} \right\} \\
& \cdot \exp \left\{ \frac{i}{\hbar} \varepsilon_s \sum_{n=0}^{N-1} \left[\frac{m}{2} \left(\frac{q_{n+1} - q_n}{\varepsilon_s} \right)^2 - V^{(3)}(q_{n+1}, t_{n+1}) \right] \right\}. \quad (88)
\end{aligned}$$

So far we have utilized all advantages proposed by the extension of the original Hilbert space $\mathcal{H}_1$ to the new one $\mathcal{H}$. Therefore we intend to remove this extension at this stage by evaluating all time integrals in (88).

First of all we note that the N delta-functions secure a set of difference equations which are consistent with (50), its specialization $f_i = f$, $f_r = 1$ in (52) and the first condition (65):

$$t_{n+1} - t_n = \varepsilon_s \cdot h'(q_{n+1}, t_{n+1})^2 \quad \text{for } n=0, 1, \dots, N-1. \quad (89)$$

Starting with the index $n=N-1$ and ceasing with $n=0$, the difference equations can be regarded as an iteration process which successively determines the values of t_n in terms of ε_s and the fixed final time t :

$$t_n = t_n(\varepsilon_s, t) \quad \text{for } n=0, 1, \dots, N-1. \quad (90)$$

The solution of the iteration procedure (89) reads

$$t - t_n = \sum_{m=n}^{N-1} \varepsilon_s \cdot h'(q_{m+1}, t_{m+1})^2 \quad \text{for } n=0, 1, \dots, N-1. \quad (91)$$

In (88) the number of delta-functions exceeds the number of time integrals by one. This has the consequence that one delta-function will eventually survive if we evaluate the $N-1$ time integrations. Choosing the Fourier representation for the remaining delta-function we obtain from (88):

$$\begin{aligned}
& G_c^{(3)}(q, t, s; q_0, t_0, 0) \\
& = \theta(s) \int_{-\infty}^{+\infty} \frac{dE}{2\pi\hbar} \exp \left\{ -\frac{i}{\hbar} E(t-t_0) \right\} \lim_{\varepsilon_s} \left\{ \prod_{n=1}^{N-1} \int dq_n \right\} \\
& \quad \cdot \left(\frac{m}{2\pi i \hbar \varepsilon_s} \right)^{\frac{N}{2}} \cdot \exp \left\{ \frac{i}{\hbar} \varepsilon_s \sum_{n=0}^{N-1} \left[\frac{m}{2} \left(\frac{q_{n+1} - q_n}{\varepsilon_s} \right)^2 \right. \right. \\
& \quad \left. \left. - V^{(3)}(q_{n+1}, t_{n+1}) + E \cdot h'(q_{n+1}, t_{n+1})^2 \right] \right\}, \quad (92)
\end{aligned}$$

where t_n is determined from (91).

If we combine the successive propagator transformations (53), (58) and (72) with (92), we can write our final result in the form of a propagator transformation formula: We have a mapping of the original causal propagator G_c with respect to the coordinate x and the time t to a new one $G_{c,E}^{(4)}$ with respect to the coordinate q and the pseudo-time s :

$$\begin{aligned}
& G_c(x, t; x_0, t_0) \\
& = F(x, t; x_0, t_0) \cdot \int_0^{+\infty} ds \int_{-\infty}^{+\infty} \frac{dE}{2\pi\hbar} \exp \left\{ -\frac{i}{\hbar} E(t-t_0) \right\} \\
& \quad \cdot G_{c,E}^{(4)}(h^{-1}(x, t), s; h^{-1}(x_0, t_0), 0). \quad (93)
\end{aligned}$$

By comparison with (85) the prefactor F is given by

$$\begin{aligned}
& F(x, t; x_0, t_0) \\
& = \sqrt{h'(h^{-1}(x, t), t) \cdot h'(h^{-1}(x_0, t_0), t_0))} \\
& \quad \cdot \exp \left\{ \frac{im}{\hbar} \cdot \left[\int^{h^{-1}(x,t)} \dot{h}(q, t) h'(q, t) dq \right. \right. \\
& \quad \left. \left. - \int^{h^{-1}(x_0, t_0)} \dot{h}(q, t_0) h'(q, t_0) dq \right] \right\}. \quad (94)
\end{aligned}$$

For the causal propagator $G_{c,E}^{(4)}$ we obtain the usual Lagrangian form of a path integral with the potential $V_E^{(4)}$. It reads in its discrete version:

$$\begin{aligned}
& G_{c,E}^{(4)}(q, s; q_0, 0) \\
& = \theta(s) \cdot \lim_{\varepsilon_s} \left\{ \prod_{n=1}^{N-1} \int dq_n \right\} \left(\frac{m}{2\pi i \hbar \varepsilon_s} \right)^{\frac{N}{2}} \\
& \quad \cdot \exp \left\{ \frac{i}{\hbar} \varepsilon_s \sum_{n=0}^{N-1} \left[\frac{m}{2} \left(\frac{q_{n+1} - q_n}{\varepsilon_s} \right)^2 - V_E^{(4)}(q_{n+1}, t_{n+1}) \right] \right\}. \quad (95)
\end{aligned}$$

From (87) the potential $V_E^{(4)}$ can be written as

$$\begin{aligned}
V_E^{(4)} & = h'^2 \left[\int h' \left\{ m \dot{h} + \frac{\partial V(h)}{\partial h} \right\} dq_{n+1} - E \right] \\
& \quad - i \hbar \dot{h}' h' + \frac{\hbar^2}{m} \left\{ \frac{3}{8} \frac{h''^2}{h'^2} - \frac{1}{4} \frac{h'''}{h'} \right\}, \quad (96)
\end{aligned}$$

where we used the abbreviation introduced in (63) and where t_n is given by (91).

The relationship (96) between the original potential V and the new one $V_E^{(4)}$ can be viewed as an expansion

into powers of Planck's constant $\hbar = 2\pi\hbar$, which stops exactly after the second order. This appears to be consistent with the fact that the Schrödinger equation itself consists only of terms up to the second order of the Planck constant $\hbar = 2\pi\hbar$.

We want to emphasize on the astonishing form of the term of order $\hbar^0$ which suggests the validity of the classical Newton equation for the arbitrary function h . The precise physical meaning of this term still remains unclear.

As has been shown during the derivation of this result, it was possible to get rid of the second of the previously mentioned problems. The second condition (85) eliminates the first order partial derivative with respect to the new coordinate in the corresponding Hamiltonian. The disadvantage of this approach is manifested in the form of the expression (96) for the potential $V_E^{(4)}$. Because of the possible imaginary part of the potential $V_E^{(4)}$ we have to deal with a transformed Hamiltonian which is in general non-hermitian.

6. Transformation of the corresponding partial differential equation

Until now the propagator transformation formula (93) has been derived by using various techniques and methods concerning path integrals. In addition to that lengthy approach we will demonstrate that the same result is obtained by implementing the transformation of time, coordinate and the wave function in the description of causal propagators by using partial differential equations. Thereby we take the opportunity to compare the different but mathematically equivalent ways of calculating causal propagators.

6.1. Equation of motion

We start by regarding the causal propagator G_c in (11) and confine ourselves to the Hamiltonian (2). Using the space representation it is possible to derive for G_c an inhomogeneous partial differential equation

$$\begin{aligned}
& \left\{ i \hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - V(x) \right\} G_c(x, t; x_0, t_0) \\
& = i \hbar \delta(x - x_0) \delta(t - t_0). \quad (97)
\end{aligned}$$

6.2. Transformation of time

The preparation for the transformation of time requires two different steps. First we introduce an additional, arbitrary function f by multiplying (97) with the quotient $f(x, t)/f(x_0, t_0)$. Because of the properties of the delta-function, the inhomogeneity is not changed by this mathematical operation:

$$\begin{aligned}
& f(x, t) \cdot \left\{ i \hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - V(x) \right\} \frac{G_c(x, t; x_0, t_0)}{f(x_0, t_0)} \\
& = i \hbar \delta(x - x_0) \delta(t - t_0). \quad (98)
\end{aligned}$$

Now we extend the dimension of the problem by introducing the pseudo-time s in the following way. We consider a partial differential equation for a new causal propagator $G_c^{(1)}$ according to

$$i\hbar \frac{\partial}{\partial s} G_c^{(1)}(x, t, s; x_0, t_0, 0) = f(x, t) \left\{ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) - i\hbar \frac{\partial}{\partial t} \right\} \cdot G_c^{(1)}(x, t, s; x_0, t_0, 0) + i\hbar \delta(x - x_0) \delta(t - t_0) \delta(s). \quad (99)$$

Performing an integration over the pseudo-time s in the interval $[0, +\infty)$ and assuming the usual boundary and initial conditions for $G_c^{(1)}$ with respect to s , we get

$$f(x, t) \left\{ i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - V(x) \right\} \cdot \int_0^{+\infty} ds G_c^{(1)}(x, t, s; x_0, t_0, 0) = i\hbar \delta(x - x_0) \delta(t - t_0). \quad (100)$$

Comparison of the two Eqs. (98) and (100) leads to a relationship between the causal propagators G_c and $G_c^{(1)}$ which is identical to (53).

6.3. Transformation of coordinate

The function h can be incorporated by performing in (99) the transformation (55) from the original coordinate x to the new one q . If we apply the properties of the delta-function, a lengthy but straight-forward calculation leads from (99) to

$$i\hbar \frac{\partial}{\partial s} G_c^{(1)}(h(q, t), t, s; h(q_0, t_0), t_0, 0) = \frac{f(h(q, t), t)}{h'(q, t)^2} \left\{ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial q^2} + \frac{\hbar^2}{2m} \left[\frac{h''(q, t)}{h'(q, t)} + \frac{2im}{\hbar} h'(q, t) \dot{h}(q, t) \right] \frac{\partial}{\partial q} + h'(q, t)^2 V(h(q, t)) - i\hbar h'(q, t)^2 \frac{\partial}{\partial t} \right\} \cdot G_c^{(1)}(h(q, t), t, s; h(q_0, t_0), t_0, 0) + i\hbar \frac{1}{h'(q_0, t_0)} \delta(q - q_0) \delta(t - t_0) \delta(s). \quad (101)$$

To simplify the notation we introduce a mapping of the causal propagator $G_c^{(1)}$ to the new one $G_c^{(2)}$ according to (58). Furthermore we impose our first condition (65) so as to reobtain in (101) the term of the free particle. The partial differential equation for the causal propagator $G_c^{(2)}$ then reads

$$i\hbar \frac{\partial}{\partial s} G_c^{(2)}(q, t, s; q_0, t_0, 0) = \left\{ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial q^2} + \frac{\hbar^2}{2m} \left[\frac{h''(q, t)}{h'(q, t)} + \frac{2im}{\hbar} h'(q, t) \dot{h}(q, t) \right] \right\}$$

$$\cdot \frac{\partial}{\partial q} + h'(q, t)^2 V(h(q, t)) - i\hbar h'(q, t)^2 \frac{\partial}{\partial t} \Big\} \cdot G_c^{(2)}(q, t, s; q_0, t_0, 0) + i\hbar \delta(q - q_0) \delta(t - t_0) \delta(s). \quad (102)$$

6.4. Transformation of the wave function

The next step consists in performing a linear transformation of the wave function according to (70) which is related to the transformation (72) of the causal propagator $G_c^{(2)}$ to $G_c^{(3)}$. We obtain from (72) and (102) a partial differential equation for the new causal propagator $G_c^{(3)}$:

$$i\hbar \frac{\partial}{\partial s} G_c^{(3)}(q, t, s; q_0, t_0, 0) = \left\{ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial q^2} - \frac{\hbar^2}{m} \left[\frac{g'(q, t)}{g(q, t)} - \frac{1}{2} \frac{h''(q, t)}{h'(q, t)} - \frac{im}{\hbar} h'(q, t) \dot{h}(q, t) \right] \cdot \frac{\partial}{\partial q} + V^{(3)}(q, t) - i\hbar h'(q, t)^2 \frac{\partial}{\partial t} \right\} \cdot G_c^{(3)}(q, t, s; q_0, t_0, 0) + i\hbar \delta(q - q_0) \delta(t - t_0) \delta(s), \quad (103)$$

where $V^{(3)}$ is given by (83). We note that (82) is the partial differential equation for the wave function $\Psi^{(3)}$ which corresponds to (103). Therefore the same considerations as in Subsects. 5.4 and 5.5 lead to the previously mentioned propagator transformation formula.

7. Special cases

The propagator transformation formula (93) describes a whole family of mappings between quantum mechanical systems. For each function $h = h(q, t)$ it is possible to transform a given quantum mechanical system with the potential V to a new one which is defined by $V_E^{(4)}$ according to (96). By choosing special functions $h = h(q, t)$ we investigate two interesting situations which can be regarded as dual in the following formal sense:

1. Special case: the function h does not depend explicitly on the time t

$$h(q, t) = h(q) \quad (104)$$

and the right hand side of the difference equations (89) is strictly a function of the new coordinate q

$$t_{n+1} - t_n = \varepsilon_s \cdot h'(q_{n+1})^2 \quad \text{for } n=0, 1, \dots, N-1. \quad (105)$$

2. Special case: if the function h is linear in the new coordinate q with

$$h(q, t) = q \cdot c(t), \quad (106)$$

the right hand side of the difference equations (89) becomes strictly a function of the time t

$$t_{n+1} - t_n = \varepsilon_s \cdot c(t_{n+1})^2 \quad \text{for } n=0, 1, \dots, N-1. \quad (107)$$

7.1. The Duru-Kleinert-propagator transformations

Because of the specialization (104) the transformed potential $V_E^{(4)}$ in (96) turns out to be independent of t_n . It therefore becomes unnecessary to insert the explicit solution of the difference equations (105) in the potential $V_E^{(4)}$. Taking into account the equations from (93) to (96), the specialization (104) leads to a family of transformations which are known as the Duru-Kleinert-propagator transformations:

$$G_c(x, t; x_0, t_0) = \sqrt{h'(h^{-1}(x)) \cdot h'(h^{-1}(x_0))} \cdot \int_0^{+\infty} ds \int_{-\infty}^{+\infty} \frac{dE}{2\pi\hbar} \cdot \exp\left\{-\frac{i}{\hbar} E(t-t_0)\right\} \cdot G_{c,E}^{(4)}(h^{-1}(x), s; h^{-1}(x_0), 0), \quad (108)$$

$$G_{c,E}^{(4)}(q, s; q_0, 0) = \theta(s) \cdot \lim_{\varepsilon_s} \left\{ \prod_{n=1}^{N-1} \int dq_n \right\} \left(\frac{m}{2\pi i \hbar \varepsilon_s} \right)^{\frac{N}{2}} \cdot \exp\left\{ \frac{i}{\hbar} \varepsilon_s \sum_{n=0}^{N-1} \left[\frac{m}{2} \left(\frac{q_{n+1} - q_n}{\varepsilon_s} \right)^2 - V_E^{(4)}(q_{n+1}) \right] \right\}, \quad (109)$$

$$V_E^{(4)}(q) = h'(q)^2 \{ V(h(q)) - E \} + \frac{\hbar^2}{m} \left\{ \frac{3}{8} \frac{h''(q)^2}{h'(q)^2} - \frac{1}{4} \frac{h'''(q)}{h'(q)} \right\}. \quad (110)$$

It is because we recover the Duru-Kleinert-propagator transformations [11] as a special case, that we choose to call our approach generalized Duru-Kleinert-propagator transformations.

7.2. Transformation to the free particle

Before we come to our second special case we shortly discuss into the question whether it is possible to perform a Duru-Kleinert-propagator transformation of a given quantum mechanical system to the free particle. This means that we are seeking for a function $h(q)$ which leads to the new potential

$$V_E^{(4)}(q) = 0. \quad (111)$$

By imposing this condition we obtain from (110) and (111) a nonlinear, ordinary differential equation of third order for the function $h = h(q)$:

$$\frac{\hbar^2}{m} \left\{ \frac{3}{8} \frac{h''(q)^2}{h'(q)^2} - \frac{1}{4} \frac{h'''(q)}{h'(q)} \right\} + h'(q)^2 \cdot V(h(q)) = h'(q)^2 \cdot E. \quad (112)$$

Because (112) does not depend explicitly on the coordinate q , it is possible to reduce the order of this ordinary differential equation. We choose as the new variable p the function $h(q)$ itself:

$$p = h(q). \quad (113)$$

Then a new function $h_1 = h_1(p)$ is identified with the first derivative of the function $h(q)$:

$$h_1(p) = h'(q). \quad (114)$$

Once the function h_1 is known, (113) and (114) determine an ordinary differential equation for the function $h(q)$:

$$h_1(h(q)) = h'(q). \quad (115)$$

From (113) and (114) one obtains the second and third derivative of the original function $h(q)$ in terms of the new function $h_1(p)$. If we choose the abbreviations

$$h'_1(p) = \frac{d}{dp} h_1(p) \quad \text{and} \quad h'(q) = \frac{d}{dq} h(q), \quad (116)$$

we have

$$\frac{h''(q)}{h'(q)} = h'_1(p) \quad \text{and} \quad \frac{h'''(q)}{h'(q)} = h'_1(p)^2 + h_1(p) h''_1(p). \quad (117)$$

The insertion of (117) in (112) finally leads to a nonlinear ordinary differential equation of second order for the new function $h_1 = h_1(p)$:

$$\frac{\hbar^2}{8m} \cdot \frac{h'_1(p)^2}{h_1(p)^2} - \frac{\hbar^2}{4m} \cdot \frac{h''_1(p)}{h_1(p)} + V(p) = E. \quad (118)$$

The further substitution

$$h_1(p) = h_2(p)^2 \quad (119)$$

transforms (118) into the time-independent Schrödinger equation of the original potential V :

$$-\frac{\hbar^2}{2m} \cdot h''_2(p) + V(p) h_2(p) = E \cdot h_2(p). \quad (120)$$

This result means that it is of no use to ask for some Duru-Kleinert-propagator transformation which directly maps a given quantum mechanical system to the free particle system. However, aided by our generalized Duru-Kleinert-propagator transformations, we demonstrate in the following how a mapping to the free particle can indeed be obtained for the special case of the harmonic oscillator. We thereby reproduce a result obtained by Inomata et al. [10].

7.3. The harmonic oscillator

Our second special case consists in the well-known quantum mechanical system of the harmonic oscillator which is defined by the potential

$$V(x) = \frac{m}{2} \omega^2 x^2. \quad (121)$$

Inserting (121) in the expression (96) for the transformed potential $V_E^{(4)}$, it appears to be adequate to fix the arbitrary function h by demanding the classical Newton equation for the harmonic oscillator:

$$m \frac{\partial^2}{\partial t^2} h(q, t) = - \frac{\partial}{\partial h(q, t)} V(h(q, t)). \quad (122)$$

Its general solution is given by (106), where the function c is determined as

$$c(t) = c_1 \cdot \cos(\omega t) + c_2 \cdot \sin(\omega t), \quad (123)$$

where c_1, c_2 are arbitrary constants.

The choice of the function h in (106) and (123) has the advantage that the transformed potential $V_E^{(4)}$ turns out to be quite simple:

$$V_E^{(4)}(q_{n+1}, t_{n+1}) = -i\hbar c(t_{n+1}) \dot{c}(t_{n+1}) - E c(t_{n+1})^2. \quad (124)$$

Using (107) has the following consequence for the discrete path integral of the new causal propagator $G_{c,E}^{(4)}$ in (95):

$$\begin{aligned} G_{c,E}^{(4)}(q, s; q_0, 0) &= \theta(s) \cdot \lim_{\varepsilon_s} \left\{ \prod_{n=1}^{N-1} \int dq_n \right\} \left(\frac{m}{2\pi i \hbar \varepsilon_s} \right)^{\frac{N}{2}} \\ &\cdot \exp \left\{ \frac{i}{\hbar} \varepsilon_s \sum_{n=0}^{N-1} \frac{m}{2} \left(\frac{q_{n+1} - q_n}{\varepsilon_s} \right)^2 \right\} \\ &\cdot \exp \left\{ \frac{i}{\hbar} \sum_{n=0}^{N-1} \left[\left(i\hbar \frac{\dot{c}(t_{n+1})}{c(t_{n+1})} - E \right) \cdot (t_{n+1} - t_n) \right] \right\}. \end{aligned} \quad (125)$$

We have to keep in mind that the quantities t_n , which depend on ε_s and on the final time t according to (90), are determined by the difference equations (107). In a continuous notation the values t_n convert in a whole function $t(\sigma)$ depending on s and t :

$$t(\sigma) = t(\sigma, s, t) \quad \text{for } \sigma \in [0, s]. \quad (126)$$

The initial value problem for the function $t(\sigma)$ is given by

$$\frac{d}{d\sigma} t(\sigma) = c(t(\sigma))^2 \quad \text{for } \sigma \in [0, s] \text{ and } t(s) = t, \quad (127)$$

where the separation of variables results in

$$\int_{t(\sigma)}^t \frac{dt(\sigma)}{c(t(\sigma))^2} = \int_{\sigma}^s d\sigma \quad \text{for } \sigma \in [0, s]. \quad (128)$$

Using the concrete expression (123) for the function c , we obtain after the trivial integration an implicit equation for the function $t(\sigma)$:

$$s - \sigma = \frac{\sin[\omega(t - t(\sigma))]}{\omega} \cdot \frac{1}{c(t) c(t(\sigma))} \quad \text{for } \sigma \in [0, s]. \quad (129)$$

Returning to the path integral (125) for the new causal propagator $G_{c,E}^{(4)}$, its continuous version reads

$$\begin{aligned} G_{c,E}^{(4)}(q, s; q_0, 0) &= \theta(s) \cdot \int_{q(0)=q_0}^{q(s)=q} \mathcal{D} q(\sigma) \exp \left\{ \frac{i}{\hbar} \int_0^s \frac{m}{2} \dot{q}(\sigma)^2 d\sigma \right\} \\ &\cdot \exp \left\{ \frac{i}{\hbar} \int_{t_0(s)}^t \left[i\hbar \frac{\dot{c}(t(\sigma))}{c(t(\sigma))} - E \right] dt(\sigma) \right\}, \end{aligned} \quad (130)$$

where $t_0(s)$ means an abbreviation for $t(0, s, t)$ from (126) and is implicitly defined in (129) for $\sigma = 0$:

$$s = \frac{\sin[\omega(t - t_0)]}{\omega} \cdot \frac{1}{c(t) c(t_0(s))}. \quad (131)$$

Explicitly performing the integration over $t(\sigma)$ the causal propagator $G_{c,E}^{(4)}$ turns out to be proportional to the propagator G_{FP} of the free particle with respect to the new coordinate q and the pseudo-time s :

$$\begin{aligned} G_{c,E}^{(4)}(q, s; q_0, 0) &= \theta(s) \cdot \exp \left\{ \frac{i}{\hbar} E(t - t_0(s)) \right\} \\ &\cdot \frac{c(t_0(s))}{c(t)} \cdot G_{FP}(q, s; q_0, 0). \end{aligned} \quad (132)$$

We suppose that the propagator G_{FP} of the free particle is known by evaluating for example the discrete form of the corresponding path integral:

$$G_{FP}(q, s; q_0, 0) = \sqrt{\frac{m}{2\pi i \hbar s}} \cdot \exp \left\{ \frac{im}{2\hbar} \cdot \frac{(q - q_0)^2}{s} \right\}. \quad (133)$$

Inserting (132) in the propagator transformation formula (93), we map the propagator of the free particle to the propagator of the harmonic oscillator

$$\begin{aligned} G_c(x, t; x_0, t_0) &= F(x, t; x_0, t_0) \int_0^{+\infty} ds \int_{-\infty}^{+\infty} \frac{dE}{2\pi \hbar} \\ &\cdot \exp \left\{ \frac{i}{\hbar} E(t_0 - t_0(s)) \right\} \cdot \theta(s) \\ &\cdot \frac{c(t_0(s))}{c(t)} \cdot G_{FP} \left(\frac{x}{c(t)}, s; \frac{x_0}{c(t_0)}, 0 \right), \end{aligned} \quad (134)$$

where the prefactor F from (94) has to be specialized in correspondence to our choice of the function h in (106):

$$\begin{aligned} F(x, t; x_0, t_0) &= \sqrt{c(t) c(t_0)} \cdot \exp \left\{ \frac{im}{2\hbar} \left[\frac{\dot{c}(t)}{c(t)} x^2 - \frac{\dot{c}(t_0)}{c(t_0)} x_0^2 \right] \right\}. \end{aligned} \quad (135)$$

In (134) we first evaluate the integral over E and obtain a delta-function:

$$\begin{aligned} G_c(x, t; x_0, t_0) &= F(x, t; x_0, t_0) \cdot \frac{c(t_0)}{c(t)} \int_0^{+\infty} ds \delta(t_0 - t_0(s)) \\ &\cdot (\theta(s) \cdot G_{FP} \left(\frac{x}{c(t)}, s; \frac{x_0}{c(t_0)}, 0 \right)). \end{aligned} \quad (136)$$

For the remaining integral over the pseudo-time s we perform a substitution of the integration variable from s to $t_0(s)$ using the relation (131):

$$\begin{aligned} G_c(x, t; x_0, t_0) &= \theta(t - t_0) \cdot F(x, t; x_0, t_0) \cdot \frac{c(t_0)}{c(t)} \int_0^{+\infty} dt_0(s) \left(-\frac{ds}{dt_0(s)} \right) \end{aligned}$$

$$\cdot \delta(t_0 - t_0(s)) \cdot G_{FP} \left(\frac{x}{c(t)}, \frac{\sin[\omega(t - t_0(s))]}{\omega} \right) \cdot \frac{1}{c(t) c(t_0(s))} \cdot \left(\frac{x_0}{c(t_0)}, 0 \right). \quad (137)$$

The derivative of $t_0(s)$ with respect to s is obtained from (128) for $\sigma = 0$:

$$\frac{d t_0(s)}{d s} = -c(t_0(s))^2. \quad (138)$$

With (138) the remaining integral in (137) over $t_0(s)$ can be calculated:

$$G_c(x, t; x_0, t_0) = \theta(t - t_0) \cdot F(x, t; x_0, t_0) \cdot \frac{1}{c(t) c(t_0)} \cdot G_{FP} \left(\frac{x}{c(t)}, \frac{\sin[\omega(t - t_0)]}{\omega} \cdot \frac{1}{c(t) c(t_0)}; \frac{x_0}{c(t_0)}, 0 \right). \quad (139)$$

Inserting the expression (133) for the propagator G_{FP} of the free particle and (135) for the prefactor F in (139), we get

$$G_c(x, t; x_0, t_0) = \theta(t - t_0) \cdot \sqrt{\frac{m\omega}{2\pi i \hbar \sin[\omega(t - t_0)]}} \cdot \exp \left\{ \frac{im}{2\hbar} \left[\frac{\dot{c}(t)}{c(t)} x^2 - \frac{\dot{c}(t_0)}{c(t_0)} x_0^2 + \frac{\omega}{\sin[\omega(t - t_0)]} \cdot c(t) c(t_0) \left(\frac{x}{c(t)} - \frac{x_0}{c(t_0)} \right)^2 \right] \right\}. \quad (140)$$

Using the explicit form (123) of the function c , we derive from (140) the well-known propagator of the harmonic oscillator. Obviously the arbitrary constants c_1, c_2 cancel and we get

Note added in proof. Meantime we became informed that there has already been some previous work to the following sections of our article: The first is concerned with Chap. 4. The idea of performing an explicit time-dependent transformation of the coordinate has already been introduced by S.N. Storchak in Phys. Lett. A **135**, 77 (1989). However, his result is slightly different from ours, because the transformation of the wave function is not considered by him. Furthermore we mention an interesting trial to the problem of Chap. 6 by G. Junker in J. Phys. A: Math. Gen. **23**, L881 (1990).

$$G_c(x, t; x_0, t_0) = \theta(t - t_0) \cdot \sqrt{\frac{m\omega}{2\pi i \hbar \sin[\omega(t - t_0)]}} \cdot \exp \left\{ \frac{im\omega}{2\hbar \sin[\omega(t - t_0)]} \cdot [(x_0^2 + x^2) \cos[\omega(t - t_0)] - 2x_0 x] \right\}. \quad (141)$$

8. Conclusions

We have presented a unification of two propagator transformations, which have up to now been considered completely independent and unrelated. In particular, our time transformation (51) formally includes both the local time transformation of Duru and Kleinert [11] and the global time transformation of Inomata et al. [10]. Whether or not there are other non-trivial examples which can exclusively be treated with our generalized Duru-Kleinert-propagator transformation, remains an open question.

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