

Dipolar Bose-Einstein condensates with weak disorder

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A homogeneous polarized dipolar Bose-Einstein condensate is considered in the presence of weak quenched disorder within mean-field theory at zero temperature. By first solving perturbatively the underlying Gross-Pitaevskii equation and then performing disorder ensemble averages for physical observables, it is shown that the anisotropy of the two-particle interaction is passed on to both the superfluid density and the sound velocity.

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Bose-Einstein condensates (BECs) in a disordered environment have been the subject of various experimental investigations in recent years. Historically, this “dirty boson” problem arose in the context of superfluid helium in Vycor glass [1]. Later, disorder either appeared naturally, for instance in magnetic wire traps [2,3], or was created artificially and controllably by using laser speckles [4,5]. Most theoretical studies focused on analyzing the Bogoliubov theory of dirty bosons for weak disorder within either the second quantization [6–9] or the replica method [10,11]. In this way, it turned out that superfluidity persists despite quenched randomness, but a depletion occurs due to the localization of tiny condensates in the respective minima of the disorder potential. So far, this localization scenario has only been analyzed for the contact interaction, as it is usually dominant for ultracold dilute Bose gases. Since the realization of atomic dipolar BECs [12] and the generation of heteronuclear molecules in the rovibrational ground state near quantum degeneracy [13], long-range and anisotropic dipole-dipole interactions have also attracted attention [14–16]. Therefore, we consider in this Rapid Communication the impact of weak disorder upon a polarized dipolar BEC at zero temperature. In particular, we will show that disorder corrections of both the superfluid density and the sound velocity yield characteristic interaction-induced anisotropies, which are not present at zero temperature in the absence of disorder.

The weakly interacting theory of dirty bosons [6–11] states that Bogoliubov quasiparticles and disorder-induced fluctuations decouple in the lowest order. This suggests a simplified approach in which the leading correction due to the presence of a random potential could be derivable from a mean-field theory. Therefore, we assume at $T = 0$ that all bosons occupy the same quantum state, for which the macroscopic wave function $\Psi(\mathbf{x})$ obeys the time-independent Gross-Pitaevskii (GP) equation

$$\left[-\frac{\hbar^2}{2m} \Delta + U(\mathbf{x}) + \int d^3x' |\Psi(\mathbf{x}')|^2 V_{\text{int}}(\mathbf{x} - \mathbf{x}') \right] \Psi(\mathbf{x}) = \mu \Psi(\mathbf{x}), \quad (1)$$

and we neglect with this from now on any impact of quantum fluctuations. Here, m denotes the particle mass, μ stands for the chemical potential, $V_{\text{int}}(\mathbf{x} - \mathbf{x}')$ represents an arbitrary

two-particle potential with inversion symmetry, and $U(\mathbf{x})$ describes the disorder potential, which is defined by its statistical properties. Denoting the disorder ensemble average according to $\langle \bullet \rangle$, a homogeneous disordered system has a vanishing first moment $\langle U(\mathbf{x}) \rangle = 0$ and the second moment is of the form $\langle U(\mathbf{x})U(\mathbf{x}') \rangle = R(\mathbf{x} - \mathbf{x}')$. The GP equation (1) represents a stochastic nonlinear partial differential equation, where the given statistics of the disorder potential $U(\mathbf{x})$ are mapped to the wave function $\Psi(\mathbf{x})$ [17]. In the case that the random potential $U(\mathbf{x})$ is small in comparison with all other energy scales, its perturbative treatment is justified. To this end, we decompose the wave function of the system according to

$$\Psi(\mathbf{x}) = \psi_0(\mathbf{x}) + \psi_1(\mathbf{x}) + \psi_2(\mathbf{x}) + \dots, \quad (2)$$

and solve the GP equation in the zeroth, first, and second order of $U(\mathbf{x})$, respectively. As the expansion (2) determines the ground state, the wave function $\Psi(\mathbf{x})$ turns out to be real. Afterwards, we determine the disorder ensemble average for both the particle density $n = \langle \Psi(\mathbf{x})^2 \rangle$ and the condensate density $n_0 = \langle \Psi(\mathbf{x}) \rangle^2$, thus, the condensate depletion results in the lowest order in

$$n - n_0 = n \int \frac{d^3k}{(2\pi)^3} \frac{R(\mathbf{k})}{[\hbar^2 \mathbf{k}^2 / 2m + 2n V_{\text{int}}(\mathbf{k})]^2} + \dots \quad (3)$$

Note that the existence of the $\mathbf{k}$ integral implies that the Fourier transform of the interaction potential $V_{\text{int}}(\mathbf{k})$ has to be strictly positive. Physically, this condensate depletion is due to the formation of fragmented condensates in the respective minima of the random potential. In order to further quantify this notion, a separate order parameter was recently proposed in Ref. [18]. It is motivated by defining the condensate density n_0 as usual as the superfluid order parameter from the off-diagonal long-range order of the one-particle density matrix [19]

$$\lim_{|\mathbf{x} - \mathbf{x}'| \rightarrow \infty} \langle \Psi(\mathbf{x}) \Psi(\mathbf{x}') \rangle = n_0. \quad (4)$$

The density of fragmented condensates q is then identified as a separate Bose-glass order parameter, similar to the Edwards-Anderson order parameter of a spin glass [20]. It can be defined

by considering the off-diagonal long-range order of the two-particle density matrix

$$\lim_{|\mathbf{x}-\mathbf{x}'| \rightarrow \infty} \langle \Psi(\mathbf{x})^2 \Psi(\mathbf{x}')^2 \rangle = (n_0 + q)^2, \quad (5)$$

as the latter has to coincide with the square of the total density n . Applying the concept of Eqs. (4) and (5) to the perturbative solution (2) of the GP equation (1) indeed yields, together with Eq. (3), the result that the density of the fragmented condensates q defined in Eq. (5) coincides in the lowest order with the condensate depletion $n - n_0$ found in Eq. (3). Thus, we conclude that the localization phenomenon for weak quenched disorder follows already from a mean-field description of the dirty boson problem. Therefore, our mean-field approach represents a simplified derivation for the disorder-induced condensate depletion (3) in comparison with the Bogoliubov theory of Refs. [6–11]. Note that, recently, disorder effects for Bogoliubov quasiparticles have been analyzed in Ref. [21].

For a polarized dipolar BEC, the particles interact via a contact interaction with strength $g = 4\pi\hbar^2 a/m$, where a denotes the s -wave scattering length, and they possess dipole moments, which are aligned along the z axis. Thus, the Fourier transform of the interaction potential is given by $V_{\text{int}}(\mathbf{k}) = g[1 + \epsilon_{\text{dd}}(3x^2 - 1)]$, where we define $x = \hat{\mathbf{k}} \cdot \hat{\mathbf{e}}_z = \cos\vartheta$ and $\epsilon_{\text{dd}} = C_{\text{dd}}/3g$ with $C_{\text{dd}} = \mu_0 \mathbf{m}^2$, in the case of magnetic dipoles with the dipole moment $\mathbf{m}$, and $C_{\text{dd}} = 4\pi \mathbf{d}^2$ for electric dipoles, where the dipole moment $\mathbf{d}$ is measured in Debyes. In the case of a spatially decaying disorder correlation $R(\mathbf{x})$, the results of the theory do not depend significantly on its shape [22]. Therefore, in what follows, we restrict ourselves to the case of a Gaussian correlation with the Fourier transform $R(\mathbf{k}) = R e^{-\sigma^2 \mathbf{k}^2/2}$, where R and σ characterize the strength and the correlation length of the disorder, respectively. With this, the condensate depletion (3) specializes to $n - n_0 = n_{\text{HM}} f(\epsilon_{\text{dd}}, \sigma/\xi) + \dots$, where the limit of pure contact interaction and delta-correlated disorder yields $n_{\text{HM}} = [m^2 R / (8\pi^{3/2} \hbar^4)] \sqrt{n/a}$ [6]. Whereas the relative interaction strength ϵ_{dd} increases the condensate depletion, the ratio of the correlation length σ and the coherence length $\xi = 1/\sqrt{8\pi n a}$ decreases it according to the function

$$f\left(\epsilon_{\text{dd}}, \frac{\sigma}{\xi}\right) = \int_0^1 dx \frac{[1 + 2\zeta(x)]e^{\zeta(x)} \text{erfc}\sqrt{\zeta(x)}}{\sqrt{3\epsilon_{\text{dd}}x^2 + 1 - \epsilon_{\text{dd}}}} - \frac{2\sigma}{\sqrt{\pi}\xi}, \quad (6)$$

with the abbreviation $\zeta(x) = \sigma^2(3\epsilon_{\text{dd}}x^2 + 1 - \epsilon_{\text{dd}})/\xi^2$ (see Fig. 1). In particular, for small ϵ_{dd} , we have $f(\epsilon_{\text{dd}}, \sigma/\xi) = A(\sigma/\xi) + B(\sigma/\xi)\epsilon_{\text{dd}}^2 + \dots$, with positive coefficients $A(\sigma/\xi)$ and $B(\sigma/\xi)$ which decrease with increasing disorder correlation length. In contrast, the condensate depletion function $f(\epsilon_{\text{dd}}, \sigma/\xi)$ diverges in the limit $\epsilon_{\text{dd}} \uparrow 1$, in accordance with the conclusion that the existence of the $\mathbf{k}$ integral in Eq. (3) is lost provided $V_{\text{int}}(\mathbf{k})$ vanishes for any $\mathbf{k}$. This indicates that our perturbative treatment breaks down provided that the dipolar interaction is strong enough relative to the contact interaction.

In order to describe a superfluid within mean-field theory, we apply a Galilean boost with wave vector $\mathbf{k}_N$ to the time-dependent GP equation and introduce a moving condensate

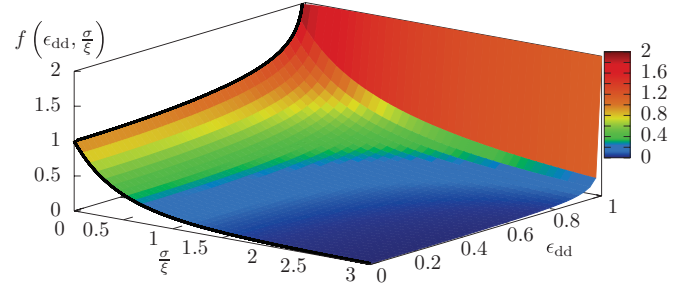


FIG. 1. (Color online) Dipolar condensate depletion function (6) with the special cases of vanishing correlation length $f(\epsilon_{\text{dd}}, 0)$ and pure contact interaction $f(0, \sigma/\xi)$ [8] indicated by solid lines.

via the ansatz $\Psi(\mathbf{x}) = \psi(\mathbf{x})e^{i\mathbf{k}_S \cdot \mathbf{x}}$ [23]. With this, we obtain in the stationary case

$$\left[-\frac{\hbar^2}{2m} \Delta - i\frac{\hbar^2}{m} \mathbf{K} \cdot \nabla + U(\mathbf{x}) - \mu_{\text{eff}} + \int d^3x' V_{\text{int}}(\mathbf{x} - \mathbf{x}') |\psi(\mathbf{x}')|^2 \right] \psi(\mathbf{x}) = 0, \quad (7)$$

where we have introduced the abbreviations $\mathbf{K} = \mathbf{k}_S - \mathbf{k}_N$ and $\mu_{\text{eff}} = \mu - \hbar^2 \mathbf{k}_S^2/2m + \hbar^2 \mathbf{k}_N \cdot \mathbf{k}_S/m$. Within a linear response, we can assume $\mathbf{k}_N$ and $\mathbf{k}_S$ to be small, so that the superfluid part of the system will move with a wave vector proportional to $\mathbf{k}_S$ and the nonsuperfluid part is coupled to $\mathbf{k}_N$. Therefore, the expansion of the total momentum $\mathbf{P} = -\int d^3x \Psi^*(\mathbf{x}) i\hbar \nabla \Psi(\mathbf{x})$ with respect to $\mathbf{k}_S$ and $\mathbf{k}_N$ yields in the disorder-averaged case $\langle \mathbf{P} \rangle = V(n_S \hbar \mathbf{k}_S + n_N \hbar \mathbf{k}_N) + \dots$, where V denotes the volume of the system and n_S and n_N represent the superfluid and nonsuperfluid density, respectively. Expanding the solution $\psi(\mathbf{x})$ of Eq. (7), which may now be complex, with respect to the disorder yields in general the result that the superfluid density turns out to be a tensor with the components

$$n_{S,ij} = n\delta_{ij} - \int \frac{d^3k}{(2\pi)^3} \frac{4nR(\mathbf{k})k_i k_j}{\mathbf{k}^2 [\hbar^2 \mathbf{k}^2/2m + 2nV_{\text{int}}(\mathbf{k})]^2} + \dots \quad (8)$$

and $n_S + n_N = n$. In the case of isotropy, i.e., $R(\mathbf{k}) = R(|\mathbf{k}|)$ and $V_{\text{int}}(\mathbf{k}) = V_{\text{int}}(|\mathbf{k}|)$, this reduces to a diagonal superfluid density $n_S = n - 4(n - n_0)/3 + \dots$, which generalizes the case of pure contact interaction [6–11].

For a dipolar BEC and Gaussian disorder correlation function, Eq. (8) yields a superfluid density that depends on the direction of the superfluid motion with respect to the orientation of the dipoles. Parallel to $\hat{\mathbf{e}}_z$, the superfluid depletion reads $n - n_{S,\parallel} = n_{\text{HM}} f_{\parallel}(\epsilon_{\text{dd}}, \sigma/\xi) + \dots$ with the function

$$f_{\parallel}\left(\epsilon_{\text{dd}}, \frac{\sigma}{\xi}\right) = 4 \int_0^1 dx \frac{x^2 [1 + 2\zeta(x)] e^{\zeta(x)} \text{erfc}\sqrt{\zeta(x)}}{\sqrt{3\epsilon_{\text{dd}}x^2 + 1 - \epsilon_{\text{dd}}}} - \frac{8\sigma}{3\sqrt{\pi}\xi}, \quad (9)$$

whereas, perpendicular to the dipoles, we have $n - n_{S,\perp} = n_{\text{HM}} f_{\perp}(\epsilon_{\text{dd}}, \sigma/\xi) + \dots$ with the function $f_{\perp}(\epsilon_{\text{dd}}, \sigma/\xi) = 2f(\epsilon_{\text{dd}}, \sigma/\xi) - f_{\parallel}(\epsilon_{\text{dd}}, \sigma/\xi)/2$. For small ϵ_{dd} , we get $f_{\parallel,\perp}(\epsilon_{\text{dd}}, \sigma/\xi) = A_{\parallel,\perp}(\sigma/\xi) + B_{\parallel,\perp}(\sigma/\xi)\epsilon_{\text{dd}} + \dots$, again with coefficients $A_{\parallel,\perp}(\sigma/\xi)$ and $B_{\parallel,\perp}(\sigma/\xi)$, whose

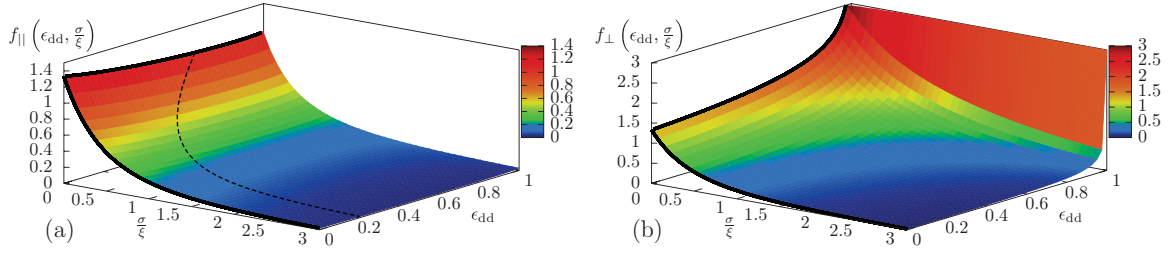


FIG. 2. (Color online) Superfluid depletion function (a) parallel and (b) perpendicular to the polarization axis of the dipoles. The solid lines indicate the special cases of vanishing correlation length and pure contact interaction, respectively. The dashed line defines the critical relative interaction strength via $f(\epsilon_{dd,crit}, \sigma/\xi) = f_{||}(\epsilon_{dd,crit}, \sigma/\xi)$.

absolute values decrease with increasing disorder correlation length. This time, however, only $f_{\perp}(\epsilon_{dd}, \sigma/\xi)$ diverges for $\epsilon_{dd} \uparrow 1$, whereas $f_{||}(\epsilon_{dd}, \sigma/\xi)$ remains finite in this limit.

Comparing Figs. 2(a) and 2(b) by taking into account the different vertical scales, we conclude that the superfluid density parallel to $\hat{e}_z$ is always less depleted than the superfluid density perpendicular to $\hat{e}_z$. This result can be qualitatively explained by considering $V_{int}(\mathbf{k})$, including the dipolar interaction, as an effective contact interaction strength for a flow with wave vector $\mathbf{k}$. This quantity is smaller than g perpendicular to $\hat{e}_z$ and larger than g parallel to $\hat{e}_z$. As the Huang-Meng depletion n_{HM} [6] for a pure contact interaction scales with the power $-1/2$ with respect to the contact interaction strength, we obtain in this picture a smaller depletion parallel to $\hat{e}_z$ than perpendicular to $\hat{e}_z$. In addition, we note that $f_{||}(0, \sigma/\xi) > f(0, \sigma/\xi)$ [8], whereas $f_{||}(\epsilon_{dd}, \sigma/\xi)$ decreases and $f(\epsilon_{dd}, \sigma/\xi)$ increases, respectively, with ϵ_{dd} for fixed σ/ξ . Therefore, a system with a sufficiently large relative interaction strength ϵ_{dd} has the astonishing property that the depletion of the parallel superfluid density is smaller than the condensate depletion. Figure 2(a) reveals in the dashed line how the critical value $\epsilon_{dd,crit}$ decreases with increasing disorder correlation length σ . The existence of $\epsilon_{dd,crit}$ represents a counterintuitive result, as particles of the fragmented BECs, which are supposed to be localized in the respective minima of the random potential, seem to contribute to the parallel superfluid motion of the system. We conclude that, due to the presence of the dipolar interaction, the locally condensed particles are only localized for a certain time scale. For longer time periods, our finding suggests that an exchange of the localized particles occurs with the nonlocalized particles, thus allowing for a superfluid density that is larger than the condensate density. This supports the finding of Ref. [18], in which such a finite localization

time for tiny BECs was calculated within a Hartree-Fock theory of dirty bosons for the special case of a pure contact interaction.

Finally, we determine the speed of sound within a hydrodynamic approach [24]. To this end we assume that, in the long-wavelength limit, the respective transport quantities are effectively disorder averaged. Thus, we consider both the Euler equation $m \partial \mathbf{v}_S / \partial t + \nabla(\mu + m \mathbf{v}_S^2 / 2) = 0$ and the continuity equation $\partial n / \partial t + \nabla \cdot \mathbf{j} = 0$ with the current density $\mathbf{j} = n_S \mathbf{v}_S + n_N \mathbf{v}_N$, where $\mathbf{v}_S$ and $\mathbf{v}_N$ denote the superfluid and boost velocities, respectively. As the normal component is pinned, we must also assume that it remains stationary. This leads to the condition $\mathbf{v}_N = \mathbf{0}$, which is reminiscent of the physically closely related problem of fourth sound in ^4He [24]. Considering small space- and time-dependent perturbations around the equilibrium values $\mathbf{v}_S = \delta \mathbf{v}_S(\mathbf{x}, t)$, $n = n_{eq} + \delta n(\mathbf{x}, t)$, $n_S = n_{S,eq} + \delta n_S(\mathbf{x}, t)$, and $\mu = \mu[n + \delta n(\mathbf{x}, t)]$ yields

$$\frac{\partial^2 \delta n(\mathbf{x}, t)}{\partial t^2} - \nabla \left[\frac{n_{S,eq}}{m} \frac{\partial \mu}{\partial n} \bigg|_{eq} \nabla \delta n(\mathbf{x}, t) \right] = 0. \quad (10)$$

With this, we obtain in the sound wave regime for the speed of sound in the direction $\hat{\mathbf{q}}$ the general result

$$\frac{c}{\sqrt{n V_{int}(\mathbf{0})/m}} = 1 + 2 \int \frac{d^3 k}{(2\pi)^3} \frac{R(\mathbf{k})}{[\hbar^2 \mathbf{k}^2 / 2m + 2n V_{int}(\mathbf{k})]^2} \times \left\{ \frac{\hbar^2 \mathbf{k}^2 V_{int}(\mathbf{k}) / 2m V_{int}(\mathbf{0})}{\hbar^2 \mathbf{k}^2 / 2m + 2n V_{int}(\mathbf{k})} - (\hat{\mathbf{q}} \hat{\mathbf{k}})^2 \right\} + \dots, \quad (11)$$

where the first term originates from the equation of state and the second term originates from the superfluid density (8). Note

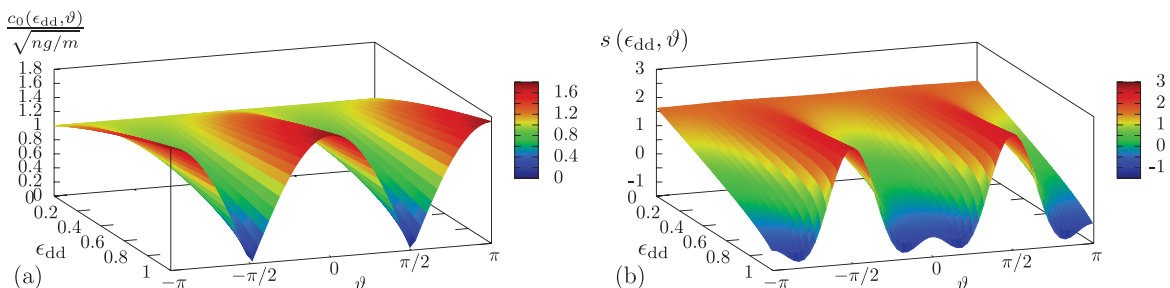


FIG. 3. (Color online) (a) Speed of sound for vanishing disorder and (b) correction term $s(\epsilon_{dd}, \vartheta)$ for nonvanishing disorder in Eq. (12).

that our hydrodynamic derivation of the speed of sound reduces for the special case of a contact interaction to an expression that has recently been obtained within an independent Green's function approach [25]. In the case of a dipolar interaction and an uncorrelated disorder $R(\mathbf{x}) = R\delta(\mathbf{x})$, we have

$$\frac{c(\epsilon_{\text{dd}}, \vartheta)}{c_0(\epsilon_{\text{dd}}, \vartheta)} = 1 + \frac{n_{\text{HMG}}/m}{2c_0^2(\epsilon_{\text{dd}}, \vartheta)} s(\epsilon_{\text{dd}}, \vartheta) + \dots, \quad (12)$$

with $c_0(\epsilon_{\text{dd}}, \vartheta) = \sqrt{ng(3\epsilon_{\text{dd}} \cos^2 \vartheta + 1 - \epsilon_{\text{dd}})/m}$ representing the speed of sound for vanishing disorder and with a dipolar function $s(\epsilon_{\text{dd}}, \vartheta)$, which generalizes the result for the pure contact interaction $s(0, \vartheta) = 5/3$ [7]. Figure 3 shows that due to weak disorder, the speed of sound has a strong dependence on both the sound direction, characterized by $\hat{\mathbf{q}} \cdot \hat{\mathbf{e}}_z = \cos \vartheta$, and the relative interaction strength ϵ_{dd} .

Finally, we conclude that the delicate interplay of dipolar interaction and weak disorder yields characteristic anisotropies for physical observables at $T = 0$. The anisotropy of the speed

of sound should be detectable with modern Bragg spectroscopy by measuring the underlying dynamic structure factor [26,27]. Even more important could be the anisotropy for the superfluidity density, as this should affect the collective excitations for harmonically trapped dipolar condensates within a random environment [28]. It is expected that these interesting effects are more pronounced for strong dipolar interactions that arise for highly magnetic atoms, such as dysprosium [29], or polar heteronuclear molecules, such as $^{40}\text{K}^{87}\text{Rb}$ [13]. To this end, it is indispensable to investigate dirty dipolar BECs in confined geometries, for instance in a harmonic trap [30] or an optical lattice [31].

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