Bose-Einstein Condensates in Compact Astrophysical Objects

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Abstract. We explore two possible applications of the theory of Bose-Einstein condensates in astrophysical contexts, one being white dwarfs and neutron stars, the other being Bose-Einstein condensates of dark matter. There is a general consensus that the conditions in these astrophysical environments allow for the formation of a Bose-Einstein condensate and thus the investigation of such scenarios are important for the determination of the physical properties of these astrophysical objects.

Keywords: Bose-Einstein condensation, astrophysics, white dwarf, neutron star, dark matter

1 Introduction

The earliest work on neutron stars has been performed by Tolman [1], as well as Oppenheimer and Volkoff [2]. They considered neutrons with an equation of state determined by Fermi statistics in a general relativistic setting and calculated the resulting stable equilibrium configuration, see also Ref. [3]. The gravitational collapse of a cloud of neutrons is counteracted by the neutron degeneracy pressure due to the Pauli exclusion principle, which leads to a prediction for the maximum masses of neutron stars of about $0.7\,M_\odot$. Observations however proved the existence of neutron stars with up to $2\,M_\odot$ [4], which is in contradiction with the limit predicted by Refs. [1, 2]. Currently there exists an abundance of different models trying to explain the observed masses of neutron stars, most of them predicting the existence of other kinds of matter in the core of a neutron star. Explanations reach from hyperons, i.e. strange baryons, over kaons and pions, both heavy mesons, to quark matter in the core, while the outer layers and crusts are supposed to be dominated by neutrons and electrons [5]. There is the general consensus that the neutrons in neutron stars are in a superfluid phase [6], i.e. they are bound in states of two neutrons, so-called Cooper pairs, and can thus effectively be treated as bosons. Investigations of typical temperatures and densities in neutron stars show that these bosons, with an effective mass of $m = 2m_n$, are in a regime in which Bose-Einstein condensation (BEC) can occur. Thus it is reasonable to investigate whether the maximum mass or

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A. Pelster and G. Wunner (eds.), *Selforganization in Complex Systems: The Past, Present, and Future of Synergetics*, Understanding Complex Systems,
DOI: 10.1007/978-3-319-27635-9_19

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other properties like the equation of state of the neutron star change under the assumption of Bose-Einstein condensated neutron pairs. This theory is known as BCS-BEC-crossover [7], and has been proven to exist in laboratory experiments on dilute ultracold quantum gases [8], but was never addressed in astrophysical settings. A second field of application for BECs in astrophysics are boson stars - either denoting the abstract concept of a star purely consisting of generic bosons [9], or using objects like white dwarfs, which contain in principle several species of particles, but are dominated by one bosonic species, like e.g. ^{4}He white dwarfs [10, 11]. Results can be compared to an abundance of existing theories on neutron stars and white dwarfs, explaining observations with varying success. Finally, some theories suggest dark matter to be bosonic and present in the form of a Bose-Einstein condensate [12–14]. This could explain some of the puzzles observed in galactic dynamics, like the rotation curves of visible matter around the center of a galaxy. Due to the completely unknown nature of dark matter, the application of BEC is yet unspecified. However, it is assumed that the conditions in dark matter halos are in principle suitable for superfluid or Bose-Einstein condensated phases of the constituent particles for some models of dark matter [15].

In the following we review the idea of applying the theory of Bose-Einstein condensation, a phenomenon occurring in cold dilute quantum gases, in the context of astrophysical compact objects. To this end we compare in Section 2 different theoretical treatments of interacting bosons in astrophysical contexts. In Section 3 we then investigate the implications in the cases of neutron stars (NSs), white dwarfs (WDs), and dark matter (DM).

2 Theoretical treatment of interacting bosons

There have been plenty of calculations in the field of Bose-Einstein condensates in astrophysical contexts, for generic boson stars as well as white dwarfs and even neutron stars. All kinds of scenarios have been considered, from Newtonian gravity to general relativistic treatments, and from non-relativistic to relativistic particle dispersions [16–18]. In the following we will introduce three approaches which differ in the assumed conditions for the investigated system.

2.1 Non-relativistic bosons in Newtonian gravity

Non-relativistic bosons are described by a condensate wave function which obeys the Gross-Pitaevskii equation, i.e. a non-linear Schrödinger type equation [16],

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{x},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + \int d^3r'\, U(\mathbf{x},\mathbf{x}')\,|\psi(\mathbf{x}',t)|^2\right]\psi(\mathbf{x},t)\,, \tag{1}$$

using contact interaction and Newtonian gravitational interaction,

$$U(\mathbf{x},\mathbf{x}') = g_0\,\delta(\mathbf{x}-\mathbf{x}') - \frac{Gm^2}{|\mathbf{x}-\mathbf{x}'|}\,, \tag{2}$$

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where m denotes the particle mass, and $g = 4\pi\hbar^2 a/m$ represents the strength of the contact interaction depending on the s-wave scattering length a.
Choosing the Madelung representation of the condensate wave function, i.e. splitting it into an amplitude and a phase according to

$$\psi(\mathbf{x},t) = \sqrt{n(\mathbf{x},t)}\; e^{iS(\mathbf{x},t)}\,, \tag{3}$$

transforms the Gross-Pitaevskii equation Eq. (1) into two coupled hydrodynamic equations for the mass density $\rho(\mathbf{x},t) = m\,n(\mathbf{x},t)$, namely the continuity equation and the Euler equation for the velocity field $\mathbf{v} = \hbar\nabla S/m$ of the fluid,

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\,\mathbf{v}) = 0\,, \tag{4a}$$

$$\rho\left[\frac{d\mathbf{v}}{dt} + (\mathbf{v}\cdot\nabla)\,\mathbf{v}\right] = -\nabla P(\rho) - \rho\,\nabla\Phi(\mathbf{x},t) - \nabla\cdot\sigma_Q\,. \tag{4b}$$

Here, the Newtonian gravitational potential $\Phi(\mathbf{x},t)$ is defined as

$$\Phi(\mathbf{x},t) = -\int d^3x'\,\frac{Gm}{|\mathbf{x}-\mathbf{x}'|}\,\rho(\mathbf{x}',t)\,, \tag{5}$$

fulfilling Poisson's equation

$$\nabla^2\Phi(\mathbf{x},t) = 4\pi G\,\rho(\mathbf{x},t)\,. \tag{6}$$

Furthermore,

$$\sigma_{ij}^{Q} = \frac{\hbar^2}{4m^2}\,\rho\,\nabla_i\nabla_j\ln\frac{\rho}{m} \tag{7}$$

denotes the so-called quantum stress tensor, which has the dimension of a pressure and represents a quantum contribution stemming from the kinetic term in the Gross-Pitaevskii equation. From the form of the equations above, the pressure can be read off as

$$P(\rho) = \frac{g}{2m^2}\,\rho^2\,, \tag{8}$$

which corresponds to a polytropic equation of state of the form $P = K\,\rho^{1+1/n}$, with the polytropic index $n = 1$.
Henceforth we will apply the Thomas-Fermi approximation in the scenario, which is justified for a system with a very large particle number and a uniform distribution of particles. Mathematically this corresponds to neglecting the quantum pressure term σ_Q. Moreover, we will assume a static configuration, and thus neglect time derivatives as well as all velocity terms. Using the identification of the pressure in Eq. (8), Eq. (4b) turns out to be

$$\nabla P = -\rho\,\nabla\Phi\,. \tag{9}$$

Here it is suitable to introduce spherical coordinates as, due to the symmetry of the problem, the angular coordinates do not occur in the calculations any more.

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Combining Eq. (6) and Eq. (9) leads, with appropriate units for radial coordinate and density, to the so-called Lane-Emden equation,

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\,\frac{d\rho}{dr}\right) = -\rho^n\,, \tag{10}$$

describing the density profile of the condensate $\rho(r)$. From this, quantities like the maximum mass of stable configurations can be calculated. Solving for the density and subsequently integrating up to the first zero of the function yields the maximum mass, which depends on the yet unspecified s-wave scattering length a of the particles [16]:

$$M_{\max} \simeq \frac{\pi}{2}\frac{\hbar c^2\sqrt{a}}{(Gm)^{3/2}}\,. \tag{11}$$

2.2 Non-relativistic neutrons in general relativistic setting

Instead of Newtonian gravity, we can also consider the laws of General Relativity [16]. With a spherically symmetric ansatz for the metric,

$$ds^2 = -e^{-\nu(r)}dt^2 + e^{\mu(r)}dr^2 + r^2 d\Omega^2\,, \tag{12}$$

and the following assumptions for the metric functions,

$$\frac{d\nu(r)}{dr} = -\left[\frac{2}{P(r)+\rho(r)c^2}\right]\frac{dP(r)}{dr}\,, \tag{13}$$

$$e^{\mu(r)} = \left[1-\frac{2GM(r)}{r}\right]^{-1}\,, \tag{14}$$

Einsteins field equations are transformed into the Tolman-Oppenheimer-Volkoff form [1, 2],

$$\frac{dP(r)}{dr} = -\frac{G\left[\rho(r)+\frac{P(r)}{c^2}\right]\left[\frac{4\pi P(r) r^3}{c^2}+M(r)\right]}{r^2\left[1-\frac{2GM(r)}{rc^2}\right]}\,, \tag{15}$$

which, together with the mass conservation equation,

$$\frac{dM(r)}{dr} = 4\pi\rho(r)\,r^2 \tag{16}$$

replace Eqs. (6) and (9). The final set of equations is then Eq. (8), (15) and (16). Processing these equations leads to a similar expression as in Eq. (10) with several additional terms which are due to general relativity. Solving for the density as a function of the radius and then integrating up to the first zero leads to the maximum mass [17]

$$M_{\max} \simeq \frac{1}{2}\frac{\hbar c^2\sqrt{a}}{(Gm)^{3/2}}\,. \tag{17}$$

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2.3 Relativistic neutrons in general relativistic setting

Now we treat relativistic neutrons, for which the equation of state changes. We will no longer use the non-relativistic Gross-Pitaevskii equation (1), but start deriving the governing equations from the action of a scalar field,

$$S = \int d^4x \sqrt{-g}\,\mathcal{L} \tag{18}$$

with the Lagrangian density

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi^* \partial_\nu \phi - \frac{1}{2} \frac{m^2 c^2}{\hbar^2} |\phi|^2 - \frac{\lambda}{4} |\phi|^4 \tag{19}$$

containing a scalar field with quartic self interaction in curved spacetime [17, 18]. A variation with respect to the complex conjugate of the scalar field ϕ^* yields the Klein-Gordon equation

$$\left(\Delta_{\mathrm{LB}} - \frac{m^2 c^2}{\hbar^2} - \lambda |\phi|^2 \right) \phi = 0\,, \tag{20}$$

where

$$\Delta_{\mathrm{LB}} = \frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu \tag{21}$$

represents the Laplace-Beltrami operator for curved spacetimes. It can be shown that with the ansatz $\phi(\mathbf{x}, t) = \exp(-imc^2 t/\hbar)\,\psi(\mathbf{x}, t)$, a spherically symmetric metric and in the non-relativistic limit the Gross-Pitaevskii equation Eq. (1) will be recovered with the identification $\lambda = 2mg/\hbar^2$.
Furthermore the energy-momentum tensor $T^{\mu\nu}$ of the scalar field follows from varying the action (18) with respect to the metric $g^{\mu\nu}$:

$$T^\mu_\nu = \frac{1}{2} g^{\mu\sigma} \left(\phi^*_{;\sigma} \phi_{;\nu} + \phi_{;\sigma} \phi^*_{;\nu} \right) - \frac{1}{2} \delta^\mu_\nu \left[g^{\kappa\sigma} \phi^*_{;\kappa} \phi_{;\sigma} + m^2 |\phi|^2 + \frac{1}{2} \lambda |\phi|^4 \right] . \tag{22}$$

Using the spherically symmetric ansatz from Eq. (12) for the metric, we end up with three equations, i.e. the Klein-Gordon equation (20), as well as the tt- and rr-components of the Einstein equations, Eqs. (9a)-(9c) in Ref. [17]. From these we obtain in analogy to above the maximum mass of the star as

$$M_{\max} \simeq 0.22 \sqrt{\frac{\lambda}{4\pi}} \frac{M_P^3}{m^2} = \frac{0.22}{\sqrt{4\pi}} \frac{\hbar c^2 \sqrt{a}}{(Gm)^{3/2}}\,, \tag{23}$$

with the Planck mass $M_P = \sqrt{\hbar c/G}$. Note that the qualitative dependence on λ has been found analytically using the Thomas-Fermi approximation, while the exact form with the coefficient 0.22 has been read off a numerical plot [17]. Investigations including thermal or quantum fluctuations have also been performed more recently in Refs. [18, 19] in a purely numerical approach.

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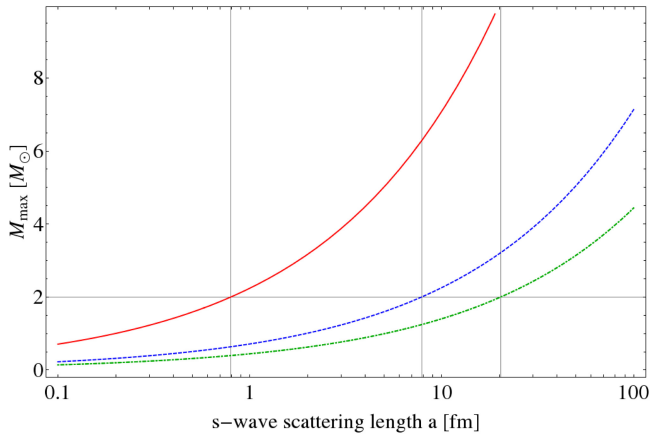


Fig. 1. Results for the predicted maximum masses of neutron stars in different treatments. Red (full line) are non-relativistic neutrons in Newtonian gravity (11), blue (dashed line) are non-relativistic neutrons in General Relativity (17), and green (dot-dashed line) are relativistic neutrons in General Relativity (23). The horizontal grid line marks the maximum observed mass of $2\,M_\odot$ [4], and the vertical grid lines denote the corresponding s-wave scattering length deduced from the respective theories.

2.4 Comparison of treatments

Even though the three models differ significantly in their physical assumptions, Fig. 1 reveals that the quantitative outcomes are rather similar and the resulting predictions all lie in the same order of magnitude. We can see that the maximum mass decreases from Newtonian to general relativistic, and from non-relativistic to relativistic treatment of neutrons. The three maximum masses are given by Eqs. (11), (17), and (23), and only differ in their numerical prefactor, which is in all three cases of the order of unity, but otherwise are proportional to $\hbar c^2\sqrt{a}/(Gm)^{3/2}$. Using the observed maximum mass of a neutron star of about $2\,M_\odot$ [4], we can infer the corresponding s-wave scattering length a of the neutrons from relations Eqs. (11), (17), and (23) and the plot in Fig. 1. The values for a are 0.8 fm, 8 fm, and 11 fm, respectively, for the three different treatments. Unfortunately astrophysical measurements do not permit the determination of the s-wave scattering length of neutrons in a neutron star, but it is surprising that the values predicted from the maximum masses are not so different from those obtained in laboratory experiments.

3 Choosing the right treatment

Some simple considerations are helpful to figure out the appropriate choice of setting for a realistic analysis. The following estimations are carried out for

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the example of a neutron star. To decide whether Newtonian gravity or General Relativity is physically appropriate, it is instructive to calculate the typical Schwarzschild radius of a neutron star from observed masses, and compare it with the observed radii of neutron stars. The Schwarzschild radius of an object is defined as

$$R_S = \frac{2GM}{c^2}\,, \tag{24}$$

and with typical masses of neutron stars of about $M \simeq 1.5\,M_\odot$, this leads to a Schwarzschild radius of about $R_S \simeq 4 \cdot 10^3\,\mathrm{m}$, which is about half of the typical radius of a neutron star of $R_{\mathrm{typ}} \simeq 10^4\,\mathrm{m}$. This clearly hints at the necessity of a general relativistic treatment.

As for the choice of treatment for the neutrons, we consider the estimated temperatures in neutron stars, $T_{\mathrm{typ}} \simeq 10^{11}\,\mathrm{K}$, and equate the thermal energy of the neutrons with their kinetic energy. Assuming for now the non-relativistic energy expression, the average velocity of the particles is given by

$$v = \sqrt{\frac{2k_B T_{\mathrm{typ}}}{m}}\,, \tag{25}$$

leading to a typical velocity of neutrons in a neutron star of $v \simeq 3 \cdot 10^7\mathrm{m/s}$, which is a velocity well in the relativistic regime. For the lower temperature of about $T_{\mathrm{typ}} \simeq 10^6\,\mathrm{K}$ in the outer regions of a neutron star however, we end up with particle velocities of around $v \simeq 9 \cdot 10^4\mathrm{m/s}$, which would justify a non-relativistic treatment. Thus, the different treatments outlined above seem to be applicable for different physical scenarios. Finally, we remark that for white dwarfs in general the Newtonian treatment of non-relativistic bosons would be sufficient, as well as for dark matter in the galactic core and halo.

4 Outlook

Plenty of calculations in the field of Bose-Einstein condensates in astrophysical contexts, ranging from generic boson stars to neutron stars and white dwarfs have been performed, and all kinds of scenarios have been considered, analytically as well as within the framework of numerical simulations. However, the majority of investigations have been carried out at zero temperature, and thus have neglected thermal fluctuations around the Bose-Einstein condensated ground state. Our estimations of temperatures and conditions in the astrophysical settings in question have shown that this assumption of zero temperature is in reality not justified, and thus thermal fluctuations would have to be taken into account. This could be accomplished in a first step by extending the zero-temperature treatment of Section 2 to a Hartree-Fock theory at finite temperature. This leads to a depletion of the condensate through the presence of thermally excited bosons. Self-consistency equations for both condensate and thermal density would change predictions of measurable quantities, such as the maximum masses shown in Fig. 1, considerably. It is expected that thermal fluctuations

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would destabilize compact astrophysical objects, resulting in a lower limit for the maximum mass.

Acknowledgments. C.G. is supported by the Erasmus Mundus Joint Doctorate Program by Grant Number 2010-1816 from the EACEA of the European Commission.

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