Rate equation description of a photon gas in a box



Kelvin Küster

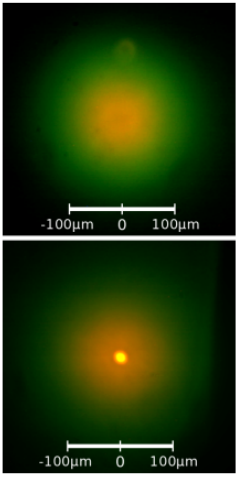
RPTU Kaiserslautern

26th of October 2026

RPTU Rheinland-Pfälzische Technische Universität Kaiserslautern Landau

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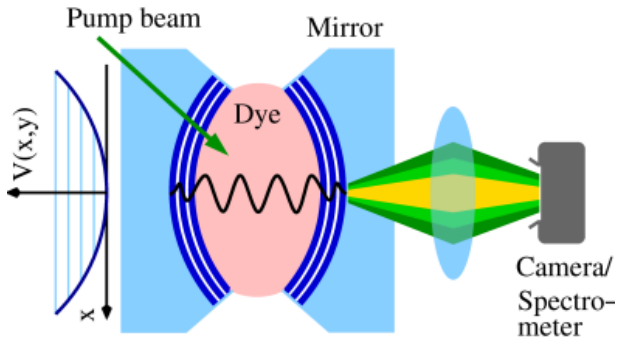
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2. Closed system
3. Open-dissipative system
4. Outlook



Klärs, Schmitt, Vewinger, and Weitz, Nature 468, 545 (2010)

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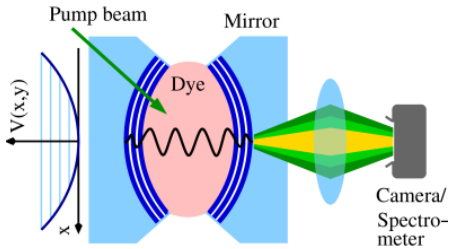
1. Experiment



Klaers, Vewinger, and Weitz, Nature Phys. 6, 512 (2010)
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1.1 Set up

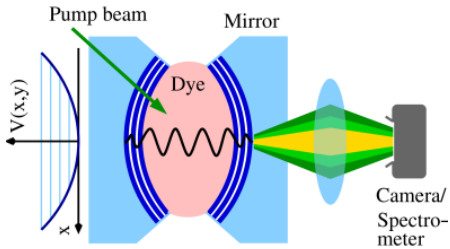
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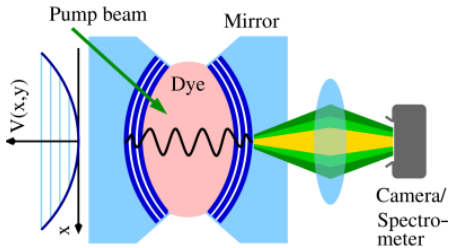
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- Mirrors create trapping potential



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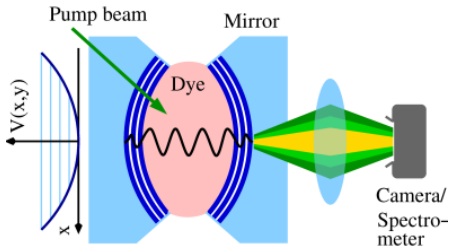
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- Optical microcavity filled with dye solution (e.g. Rhodamine 6G)
- Mirrors create trapping potential
- Dye molecules act as photon bath
- Photons leaving the cavity are observed with camera or spectrometer



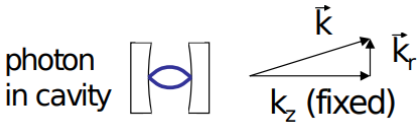
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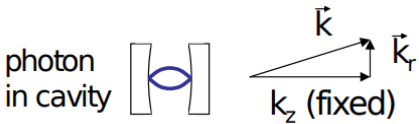


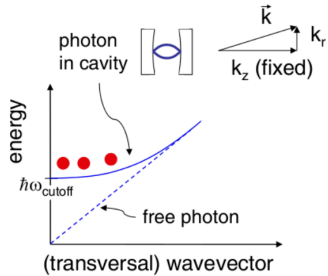
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- Cavity introduces quadratic dispersion: $E_{\vec{k}} = \underbrace{Mc^2}_{=E_{\mathrm{cut}}} + \frac{\hbar^2 (k_x^2 + k_y^2)}{2M} + \ldots$

  $\rightarrow M_{\mathrm{eff}} \approx 7.8 \cdot 10^{-36}\ \mathrm{kg} \approx 10^{-5} M_e$





M. Weitz (2018)

1.2 Conditions for photon BEC

- Multiple cycles of absorption and re-emission
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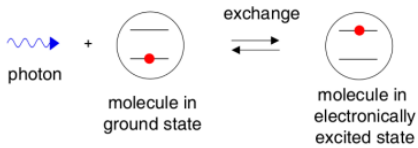
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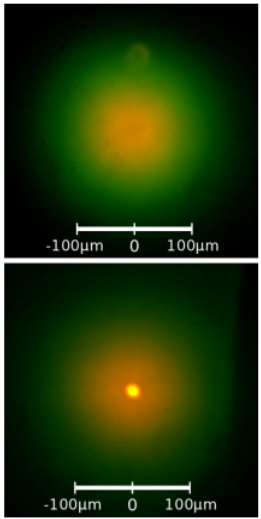
Thermal deBroglie wavelength:

$$\lambda_{\mathrm{th}} = \sqrt{\frac{2\pi\hbar^2}{M k_{\mathrm{B}} T}}$$

Inter particle distance: $d = (N/V)^{-1/3}$

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2.1 Statistical mechanics

- Periodic boundary conditions: $k_i = 2\pi \frac{n_i}{L},\ n_i \in \mathbb{Z}$

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- Dirichlet boundary conditions: $k_i = \pi \frac{n_i}{L},\ n_i \in \mathbb{N}$

2.1 Statistical mechanics

- Periodic boundary conditions: $k_i = 2\pi \frac{n_i}{L},\ n_i \in \mathbb{Z}$
- Dirichlet boundary conditions: $k_i = \pi \frac{n_i}{L},\ n_i \in \mathbb{N}$
- Bose-Einstein distribution: $N = \sum_{\vec{n}} \frac{1}{e^{\beta(E_{\vec{n}}-\mu)}-1} = N_{\mathrm{g}} + \sum_{\vec{n}\neq\vec{n}_{\mathrm{g}}} \frac{1}{e^{\beta(E_{\vec{n}}-\mu)}-1}$ $(\beta = \frac{1}{k_{\mathrm{B}}T})$

2.2 Rate equations: closed system

> Rate equation for closed system in steady state:

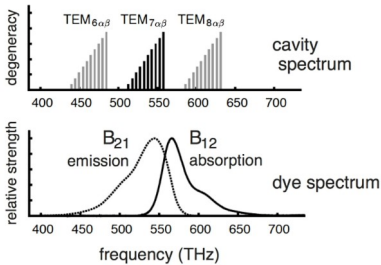
$$\dot{N}_{\vec{n}} = 0 = \underbrace{B_{21}(E_{\vec{n}})M_2N(E_{\vec{n}}) + A_{21}(E_{\vec{n}})M_2}_{2\to1} - \underbrace{B_{12}(E_{\vec{n}})M_1N(E_{\vec{n}})}_{1\to2}$$

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- Einstein relation: $B_{21}(E_{\vec{n}}) = A_{21}(E_{\vec{n}})$
$$\Rightarrow N(E_{\vec{n}}) = \frac{1}{\frac{B_{12}(E_{\vec{n}})M_1}{B_{21}(E_{\vec{n}})M_2} - 1}$$

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- Kennard-Stepanov: $B_{21}(E_{\vec{n}}) = e^{-\frac{E_{\vec{n}} - E_{\mathrm{ZPL}}}{k_{\mathrm{B}}T}} B_{12}(E_{\vec{n}})$



J. Klaers, J. Schmitt, T. Damm, F. Vewinger, M. Weitz. Applied Physics B, 105:17–33, 2011.

2.2 Rate equations: closed system

$\Rightarrow N(E_{\vec{n}}) = \frac{1}{e^{\beta(E_{\vec{n}}-\mu)}-1}$ $\rightarrow$ Rate equations lead to same distribution!

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$>$ Chemical potential has to be defined as $\mu(M_1, M_2) = E_{\mathrm{ZPL}} - \frac{1}{\beta}\ln\left(\frac{M_1}{M_2}\right)$

2.3 Thermodynamic limit

- Dispersion:
  $E_{\vec{k}} = \frac{\hbar^2}{2M}|\vec{k}|^2 = \frac{2\hbar^2\pi^2}{ML^2}(n_x^2 + n_y^2)$

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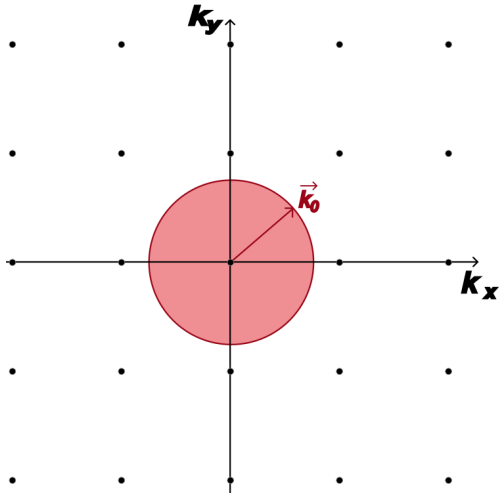
$$N_{\mathrm{th}} = \sum_{\vec{n}\neq\vec{n}_g} \frac{1}{e^{\beta(E_{\vec{n}}-\tilde{\mu})}-1}$$

with $\tilde{\mu} = \mu - E_{\mathrm{cut}}$

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  with $\tilde{\mu} = \mu - E_{\mathrm{cut}}$
- Cut out disc in phase space due to ground state
  $\rightarrow$ infra-red cut-off



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2.3 Thermodynamic limit

> Euler-MacLaurin: $\sum_{n_x,n_y} f(n_x,n_y) \approx \int\int f(n_x,n_y)\mathrm{d}n_x\mathrm{d}n_y$

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- Shift to polar coordinates (k,ϕ) for cutout

$$\Rightarrow N_{\mathrm{th}} \approx \frac{L^2}{4\pi^2}\int_0^{2\pi}\mathrm{d}\phi\int_{k_0}^{\infty}\mathrm{d}k\frac{k}{e^{\frac{\hbar^2k^2}{2Mk_{\mathrm{B}}T}-\frac{\tilde{\mu}}{k_{\mathrm{B}}T}}-1}$$

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- Substitute $x=\hbar^2k^2/(2Mk_{\mathrm{B}}T)$ and integrate
- Result:

$$\boxed{N_{\mathrm{th}} = -\frac{L^2}{\lambda_{\mathrm{th}}^2}\ln\left(1-e^{\frac{\tilde{\mu}}{k_{\mathrm{B}}T}-\frac{\lambda_{\mathrm{th}}^2}{L^2}c_0}\right)}$$

$$\rightarrow c_0 = \frac{k_0^2L^2}{4\pi} \geq 0$$

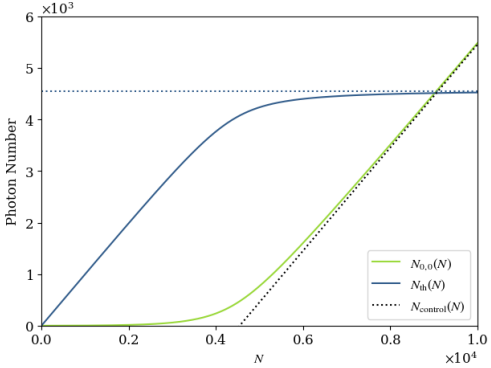
2.4 Photon occupation (periodic boundary conditions)

> Equation of state: $N = \sum_{n_x, n_y = -\infty}^{\infty} \frac{1}{e^{\pi \frac{\lambda_{\mathrm{th}}^2}{L^2}(n_x^2 + n_y^2) - \frac{\tilde{\mu}}{k_{\mathrm{B}} T}} - 1}$

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- Calculate N and $N_{0,0} = \frac{1}{e^{-\frac{\tilde{\mu}}{k_{\mathrm{B}} T}} - 1}$ numerically

  $\rightarrow N_{\mathrm{th}} = N - N_{0,0}$

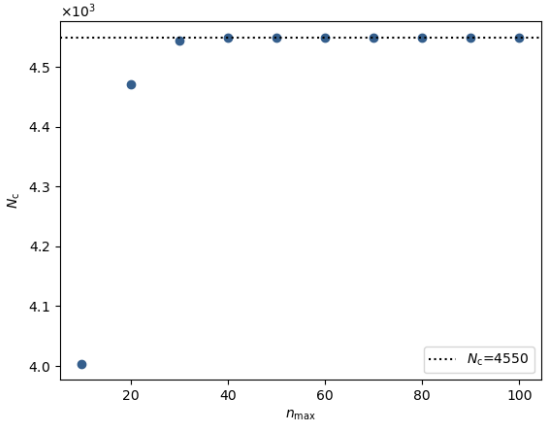


$L = 40\,\mu\mathrm{m}$

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2.5 Convergence of periodic numerics

> How many modes need to be considered for convergence?
> $\rightarrow$ deduce via critical particle number!



$L = 40\,\mu\text{m}$

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2.6 Unified description for c_0

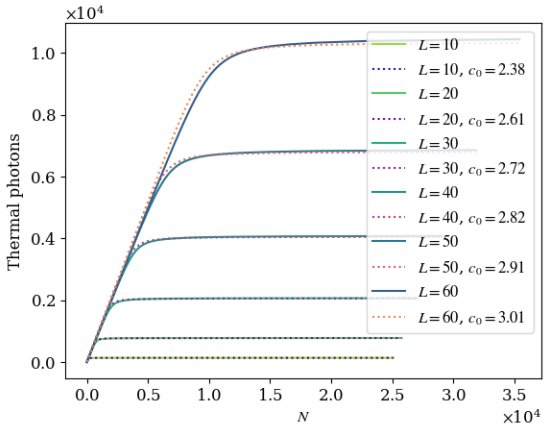
- Reminder: $N_{\mathrm{th}}^{\mathrm{ana}} = -\frac{L^2}{\lambda_{\mathrm{th}}^2} \ln\left(1 - e^{\frac{\tilde{\mu} - E_{\mathrm{g}}}{k_{\mathrm{B}} T} - \frac{\lambda_{\mathrm{th}}^2}{L^2} c_0}\right)$
- Determine $N_{\mathrm{c}}^{\mathrm{ana}}$:

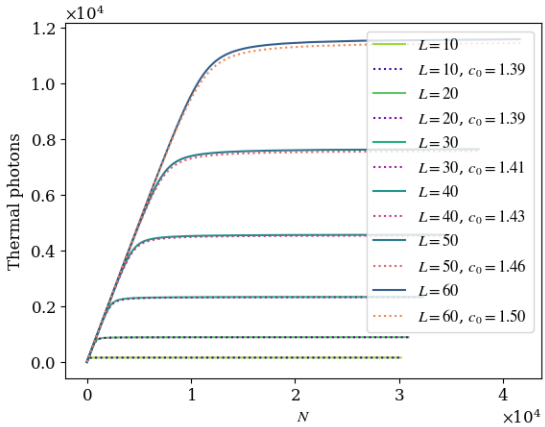
$$N_{\mathrm{c}} = \lim_{\tilde{\mu} - E_{\mathrm{g}} \uparrow 0} N_{\mathrm{th}}^{\mathrm{ana}} = -\frac{L^2}{\lambda_{\mathrm{th}}^2} \ln\left(1 - e^{-\frac{\lambda_{\mathrm{th}}^2}{L^2} c_0}\right)$$

$$\Rightarrow c_0 = -\frac{L^2}{\lambda_{\mathrm{th}}^2} \ln\left(1 - e^{-\frac{\lambda_{\mathrm{th}}^2}{L^2} N_{\mathrm{c}}}\right)$$

- Determine N_{c} numerically to calculate c_0, then compare graphs

2.7 Dirichlet compared to periodic numerics



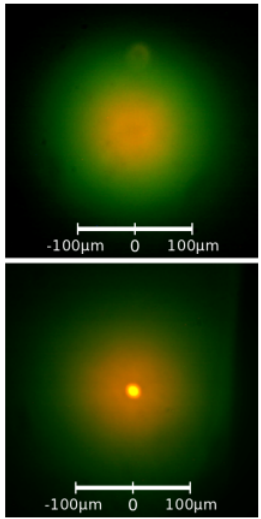


> Dirichlet boundary conditions

> Periodic boundary conditions

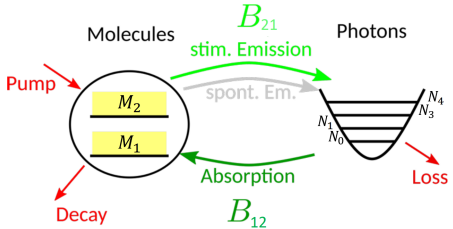
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3.1 Rate equations



1. $\dot{N}_{n_x,n_y} = [B_{21}(E_{n_x,n_y})M_2 - B_{12}(E_{n_x,n_y})M_1 - \Gamma_\mathrm{c}(E_{n_x,n_y})]N_{n_x,n_y} + B_{21}(E_{n_x,n_y})M_2$
2. $\dot{M}_2 = pM_1 - \Gamma_\mathrm{nr}M_2 - \sum_{n_x,n_y} \left\{[B_{21}(E_{n_x,n_y})M_2 - B_{12}(E_{n_x,n_y})M_1]N_{n_x,n_y} + B_{21}(E_{n_x,n_y})M_2\right\}$
3. $M = M_1 + M_2$

> New parameters: cavity losses Γ_c, non-radiative losses Γ_nr, external pumping rate p.

> Reminder: $B_{21}(E_{n_x,n_y}) = A_{21}(E_{n_x,n_y})$

P. Kirton and J. Keeling, PRA 91, 033826 (2015); M. Radonjić et alii, NJP 20, 055014 (2018); E. Stein, PhD thesis (2022); J. Krauß, E. Stein, A. Pelster, EPJ, 235:4311 (2026)

3.1.1 Rate equations in steady state

- $\dot{N} = 0 = [B_{21}(E_{n_x,n_y})M_2 - B_{12}(E_{n_x,n_y})M_1 - \Gamma_c(E_{n_x,n_y})]N_{n_x,n_y} + B_{21}(E_{n_x,n_y})M_2$
  $\Rightarrow N = \sum_{n_x,n_y} \frac{1}{e^{\beta(E_{n_x,n_y}-\mu)}-1+\frac{\Gamma_c(E_{n_x,n_y})}{B_{21}(E_{n_x,n_y})M_2}}$
  with $\mu(M_1, M_2) = E_{\mathrm{ZPL}} - \frac{1}{\beta}\ln\left(\frac{M_1}{M_2}\right)$

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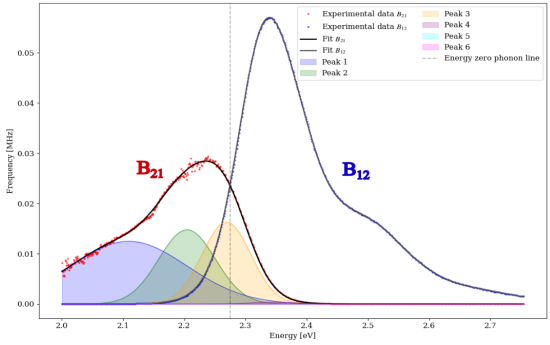
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  with $\mu(M_1, M_2) = E_\mathrm{ZPL} - \frac{1}{\beta}\ln\left(\frac{M_1}{M_2}\right)$
- $\dot{M}_2 = \dot{M}_1 = 0$

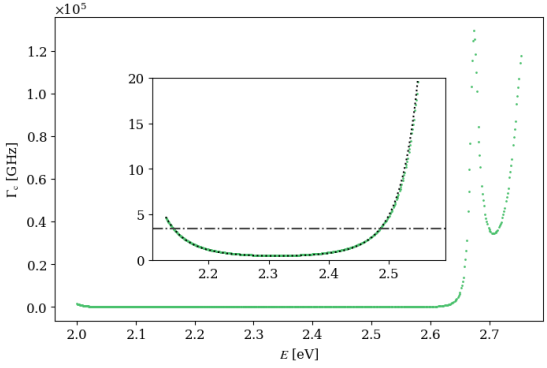
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  with $\mu(M_1, M_2) = E_\mathrm{ZPL} - \frac{1}{\beta}\ln\left(\frac{M_1}{M_2}\right)$
- $\dot{M}_2 = \dot{M}_1 = 0$
- Issue: Einstein coefficients and cavity losses are energy dependent! $\rightarrow$ Fit experimental data!

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3.2 Experimental values





> Fit emission: $B_{21}(E) = \sum_{i=1}^{n} A_i e^{-\frac{(E-\Theta_i)^2}{2\sigma_i^2}}$

$\rightarrow$ Kennard-Stepanov yields absorption

> Fit cavity losses: $\Gamma_c(E) = e^{a(E-b)^2+c}$

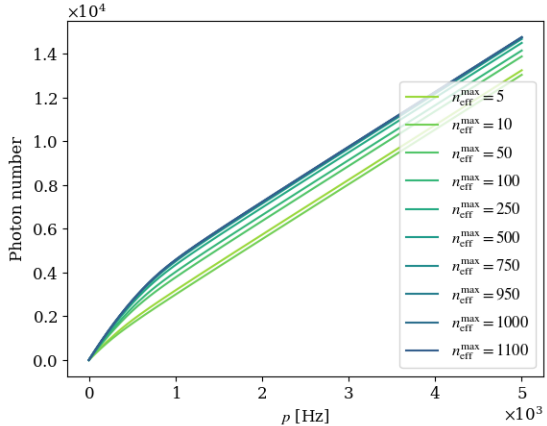
J. Krauß, E. Stein, A. Pelster, EPJ, 235:4311 (2026)

zenodo.org/records/10852936, J. Schmitt, M. Weitz, J. Klaers

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3.3 Convergence: Dirichlet boundary conditions

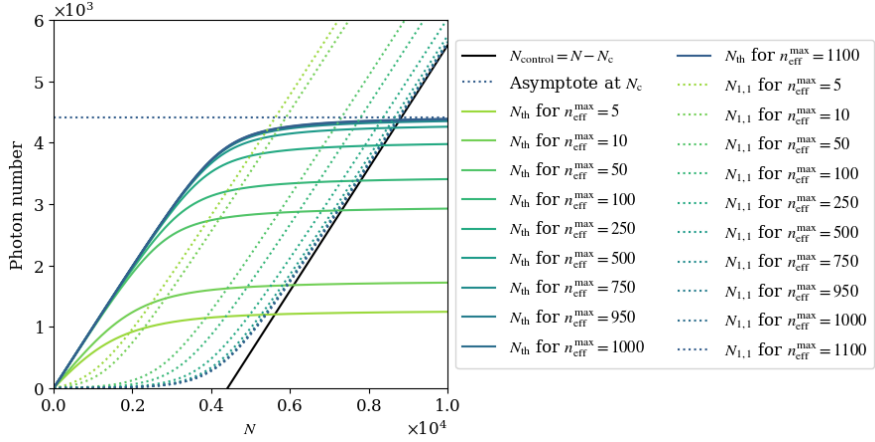
> Solve for starting values $p = 0, M = M_1, M_2 = 0$



> $L = 40\,\mu\text{m}, \Gamma_{\text{nr}} = 0, \Gamma_{\text{c}} = 4\,\text{GHz}$

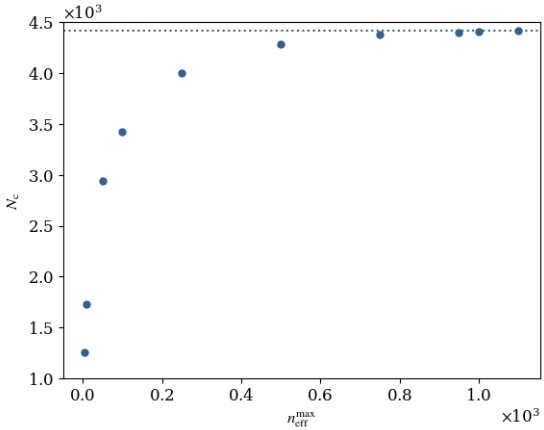
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3.3 Convergence: Dirichlet boundary conditions

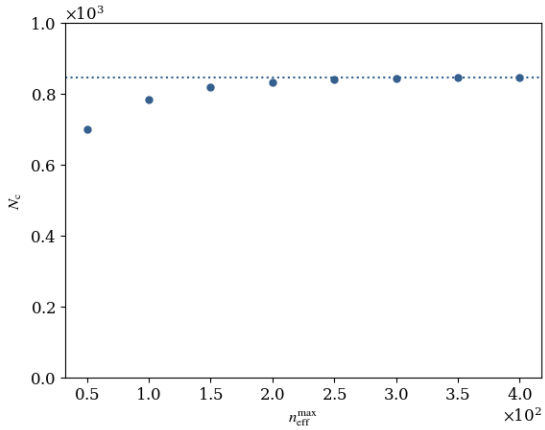


> $L = 40\,\mu\text{m}, \Gamma_{\text{nr}} = 0, \Gamma_{\text{c}} = 4\,\text{GHz}$

3.3 Convergence: Dirichlet boundary conditions



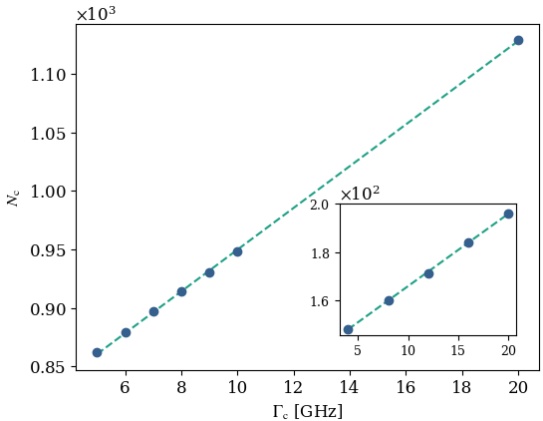
> $L = 40\,\mu\mathrm{m}, \Gamma_{\mathrm{nr}} = 0, \Gamma_{\mathrm{c}} = 4\,\mathrm{GHz}$

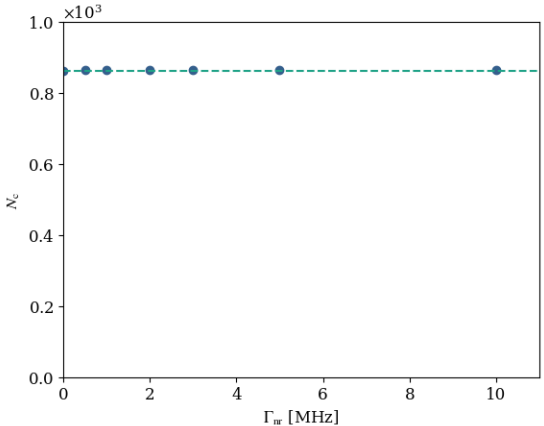


> $L = 20\,\mu\mathrm{m}, \Gamma_{\mathrm{nr}} = 0, \Gamma_{\mathrm{c}} = 4\,\mathrm{GHz}$

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3.4 Sensitivity to losses



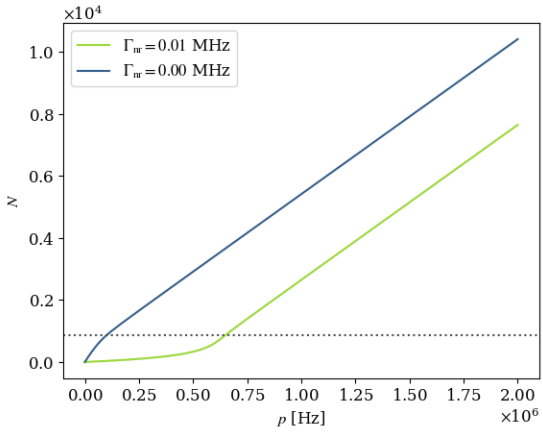


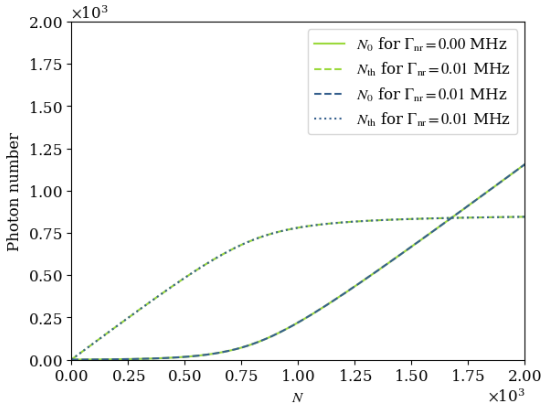
> $L = 40\,\mu\text{m}$ (inset: $L = 20\,\mu\text{m}$), $\Gamma_{\text{nr}} = 0$

> $L = 20\,\mu\text{m}, \Gamma_{\text{c}} = 5\,\text{GHz}$

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3.4 Sensitivity to losses

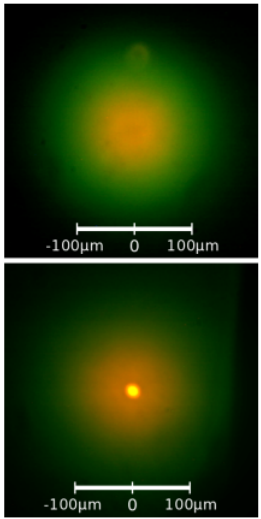




> $L = 20\,\mu\text{m}, \Gamma_c = 5\,\text{GHz}$

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1. Experiment
2. Closed system
3. Open-dissipative system
4. Outlook

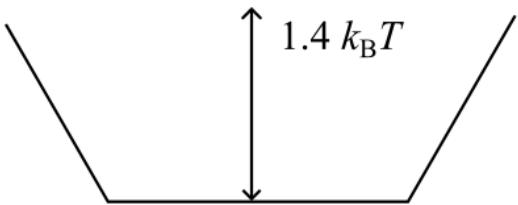


4.1 Outlook I.

Closed system

> Analytical derivation of c_0?

Open-dissipative system

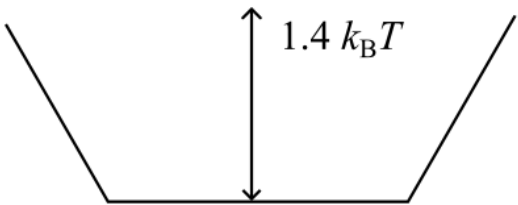


4.1 Outlook I.

Closed system

- Analytical derivation of c_0?
- More accurate description of box
  - → Slope
  - → Trap depth

Open-dissipative system



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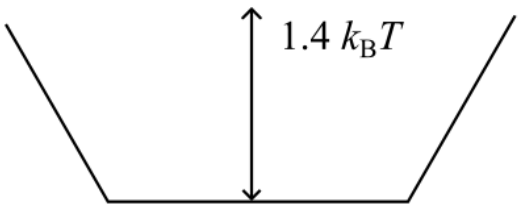
4.1 Outlook I.

Closed system

- Analytical derivation of c_0?
- More accurate description of box
  → Slope
  → Trap depth

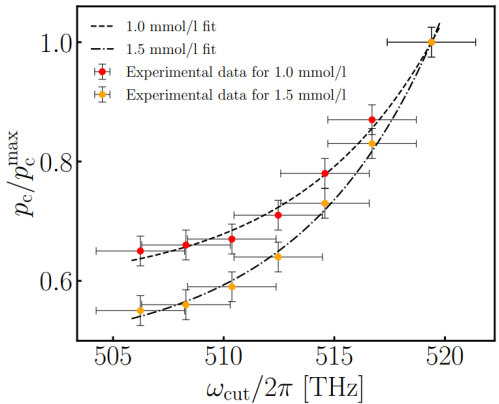
Open-dissipative system

- Extension of analytical treatment to open-dissipative system?



4.2 Outlook II.: Experiment?

> Experimental verification analogue to harmonic trap?



J. Krauß, S. Enns
(private communication)

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