Interacting Fermi Gas in a Spherical Shell

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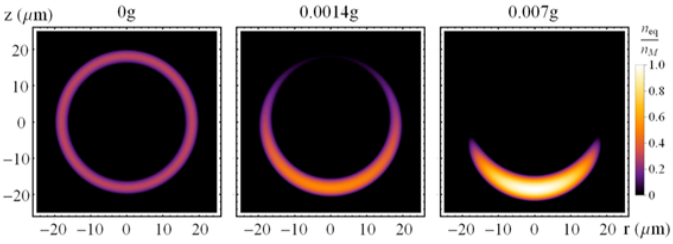
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Overview

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3. Curvature and Interaction Parameters
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1 Motivation

How to get quantum gases in spherical shell?



Numerical density profiles of a $^{87}\mathrm{Rb}$ condensate shell under different gravitational strengths. Credit: Karmela Padavic-Callaghan, Illinois Physics.

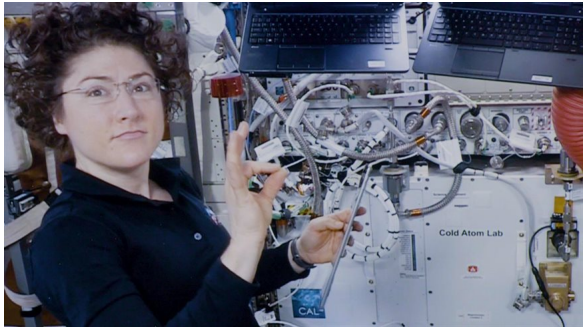
Goal: full shell geometry.

Problem: gravity pulls atoms downward.

Effect: density accumulates near the bottom.

Need: microgravity to access nearly uniform S^2 gases.

Cold Atom Lab: microgravity platform



NASA Cold Atom Lab. Credit: Karmela Padavic-Callaghan, Illinois Physics.

Bubble traps: shell-shaped potential minimum.

Microgravity: suppresses gravitational sag.

Physics platform: curved, closed, finite-size quantum gases.

This project: interacting fermions on $S^2(R)$.

From bubble traps to a spherical surface

Bubble trap: atoms confined near a shell-shaped minimum.

Strong radial confinement: radial motion is frozen.

Low-energy motion: mainly along the shell surface.

Effective geometry: two-dimensional quantum gas on $S^2(R)$.

Question of this project

How do finite curvature and the intrinsic spherical two-body bound state modify the BCS–BEC crossover of an interacting Fermi gas on $S^2(R)$?

2 In the Literature

Literature Review on spherical-shell BCS theory

He, Guo and Chien:

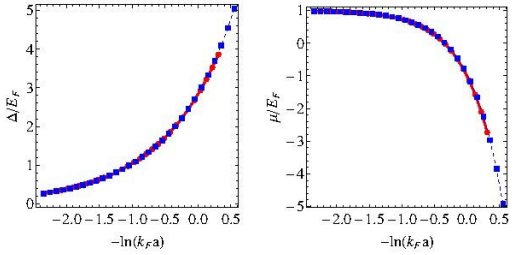
$$E_l = \frac{\hbar^2 l(l+1)}{2MR^2}, \qquad d_l = 2l+1.$$

Two-body input:

$$E_{\mathrm{bind}}^{\mathbb{R}^2} = -\frac{\hbar^2}{Ma_{2\mathrm{D}}^2}.$$

Thermodynamic limit:

$$\frac{1}{4\pi R^2}\sum_l \quad \longrightarrow \quad \int dl.$$



Results from flat space model (blue) and thermodynamic-limit sphere model (red) coincide.

Present treatment:

- discrete finite-$k_{\mathrm{F}}R$ sums,
- intrinsic spherical two-body binding energy.

3 Curvature and Interaction Parameters

Curvature and finite-size parameter

For a spin-balanced closed shell,

$$N = 2\sum_{l=0}^{l_{\mathrm{F}}}(2l+1) = 2(l_{\mathrm{F}}+1)^2.$$

The Fermi energy is

$$E_{\mathrm{F}} = E_{l_{\mathrm{F}}} = \frac{\hbar^2 l_{\mathrm{F}}(l_{\mathrm{F}}+1)}{2MR^2}.$$

Matching this with

$$E_{\mathrm{F}} = \frac{\hbar^2 k_{\mathrm{F}}^2}{2M}$$

gives

$$\boxed{\frac{1}{k_{\mathrm{F}}R} = \frac{1}{\sqrt{l_{\mathrm{F}}(l_{\mathrm{F}}+1)}}.}$$

Meaning of $1/k_{\mathrm{F}}R$

$1/(k_{\mathrm{F}}R)$: measures finite size and curvature of sphere relative to the Fermi wavelength.

For large particle number,

$$\frac{1}{k_{\mathrm{F}}R} \sim \sqrt{\frac{2}{N}}.$$

Thus,

$$\frac{1}{k_{\mathrm{F}}R} \to 0$$

is both the large-system limit and the flat-space limit.

The sphere is now characterized by this curvature parameter $1/(k_{\mathrm{F}}R)$.

Interaction coordinate

We use the same crossover coordinate as in flat 2D:

$$\boxed{x = -\log(k_{\mathrm{F}} a_{2\mathrm{D}}).}$$

However, the intrinsic spherical two-body correction depends on

$$\frac{a_{2\mathrm{D}}}{R}.$$

Using $k_{\mathrm{F}} a_{2\mathrm{D}} = e^{-x}$,

$$\boxed{\frac{a_{2\mathrm{D}}}{R} = \frac{e^{-x}}{k_{\mathrm{F}} R}.}$$

At fixed x, changing $k_{\mathrm{F}} R$ changes $a_{2\mathrm{D}}/R$. Therefore the fermionic spherical system depends on both x and $1/(k_{\mathrm{F}} R)$.

4 Spherical Binding Energy

Intrinsic spherical two-body input

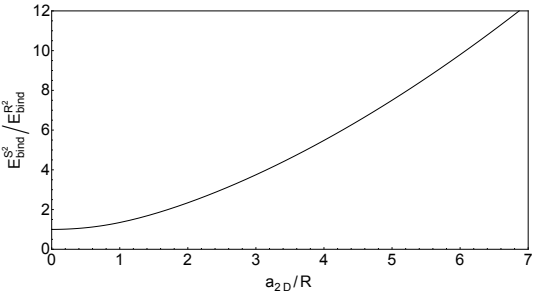
For a contact interaction on $S^2(R)$:

$$\boxed{E_{\mathrm{bind}}^{S^2(R)} = \eta\left(\frac{a_{2\mathrm{D}}}{R}\right) E_{\mathrm{bind}}^{\mathbb{R}^2}.}$$

η is the curvature-dependent correction obtained from the renormalized two-body problem on the sphere.

In the flat limit,

$$\frac{a_{2\mathrm{D}}}{R} \to 0, \qquad \eta \to 1.$$



Spherical correction to the two-body binding energy.

Binding-energy correction

In flat 2D,

$$\frac{|E_{\mathrm{bind}}^{\mathbb{R}^2}|}{E_{\mathrm{F}}} = 2e^{2x}.$$

On the sphere,

$$\boxed{\frac{|E_{\mathrm{bind}}^{S^2(R)}|}{E_{\mathrm{F}}} = 2e^{2x}\eta\left(\frac{e^{-x}}{k_{\mathrm{F}}R}\right).}$$

Thus the same interaction coordinate x give different binding energies at different values of $k_{\mathrm{F}}R$.

$$\frac{a_{2\mathrm{D}}}{R} = \frac{e^{-x}}{k_{\mathrm{F}}R}$$

$$\frac{a_{2\mathrm{D}}}{R} \to 0 \quad \Longrightarrow \quad E_{\mathrm{bind}}^{S^2(R)} \to E_{\mathrm{bind}}^{\mathbb{R}^2}.$$

5 Mean-Field Theory

Mean-field Hamiltonian

The BCS mean-field Hamiltonian on the sphere is

$$\hat{H}_{\mathrm{BCS}}^{(\mathrm{MF})} = \sum_{l,m} \left[\sum_{\sigma \in \{\uparrow,\downarrow\}} (E_l - \mu) \hat{a}_{l,m,\sigma}^{\dagger} \hat{a}_{l,m,\sigma} + \Delta^{(\mathrm{MF})} \hat{a}_{l,m,\uparrow}^{\dagger} \hat{a}_{l,-m,\downarrow}^{\dagger} + \Delta^{(\mathrm{MF})*} \hat{a}_{l,-m,\downarrow} \hat{a}_{l,m,\uparrow} \right] - \frac{4\pi}{g} |\Delta^{(\mathrm{MF})}|^2.$$

Bogoliubov transformation:

$$\begin{bmatrix} \hat{\alpha}_{l,m} \\ \hat{\beta}_{l,m}^{\dagger} \end{bmatrix} = U_{\mathrm{B}}(l,m) \begin{bmatrix} \hat{a}_{l,m,\uparrow} \\ \hat{a}_{l,-m,\downarrow}^{\dagger} \end{bmatrix}.$$

Diagonal form:

$$\boxed{\hat{H}_{\mathrm{BCS}}^{(\mathrm{MF})} = \sum_{l,m} E_{\mathrm{B}}(l) \left(\hat{\alpha}_{l,m}^{\dagger} \hat{\alpha}_{l,m} + \hat{\beta}_{l,m}^{\dagger} \hat{\beta}_{l,m} \right) + \sum_{l,m} [(E_l - \mu) - E_{\mathrm{B}}(l)] - \frac{4\pi}{g} |\Delta^{(\mathrm{MF})}|^2}$$

$$\boxed{E_{\mathrm{B}}(l) = \sqrt{(E_l - \mu)^2 + |\Delta^{(\mathrm{MF})}|^2}.}$$

Density equation

The density follows from

$$n = -\frac{1}{4\pi R^2}\frac{\partial \mathcal{F}_{\mathrm{BCS}}^{(\mathrm{MF})}}{\partial \mu}.$$

This gives

$$\boxed{n = \frac{1}{4\pi R^2}\sum_l (2l+1)\left[1 - \frac{E_l - \mu}{E_{\mathrm{B}}(l)}\tanh\left(\frac{\beta E_{\mathrm{B}}(l)}{2}\right)\right].}$$

The factor $4\pi R^2$ is the physical area of the sphere.

Order-parameter equation

The order parameter is determined by the extremization condition

$$\frac{\partial \mathcal{F}_{\mathrm{BCS}}^{(\mathrm{MF})}}{\partial |\Delta^{(\mathrm{MF})}|} = 0.$$

This gives the bare equation

$$0 = \frac{8\pi}{g} + \sum_{l} (2l+1) \frac{\tanh\left(\frac{\beta E_{\mathrm{B}}(l)}{2}\right)}{E_{\mathrm{B}}(l)}.$$

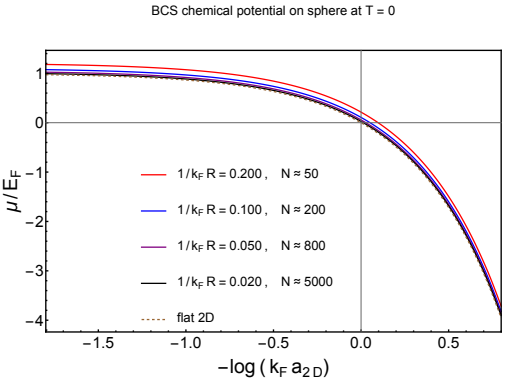
Using the spherical two-body bound-state identity

$$\frac{1}{g} = \frac{1}{4\pi} \sum_{l} \frac{2l+1}{E_{\mathrm{bind}}^{S^2(R)} - 2E_l},$$

$$\boxed{0 = \sum_{l} (2l+1) \left[-\frac{1}{|E_{\mathrm{bind}}^{S^2(R)}| + 2E_l} + \frac{1}{2E_{\mathrm{B}}(l)} \tanh\left(\frac{\beta E_{\mathrm{B}}(l)}{2}\right) \right].}$$

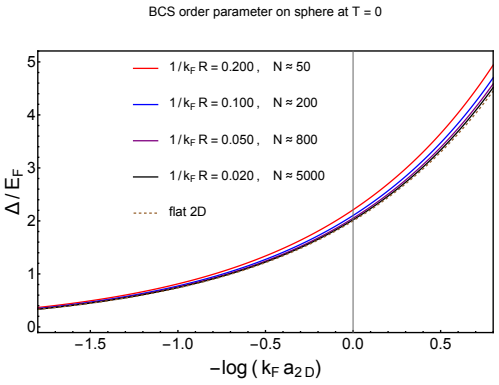
6 Results

Chemical potential at $T = 0$



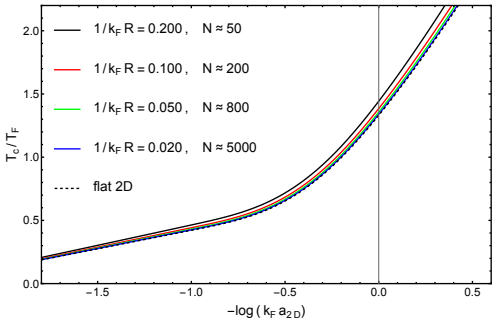
Finite curvature shifts the chemical potential away from the flat-2D curve. The curves approach flat 2D as $1/(k_{\mathrm{F}} R) \to 0$.

Order parameter at $T = 0$



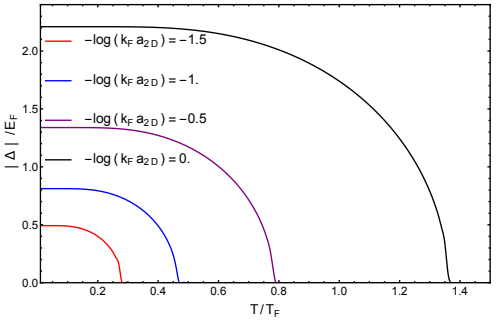
The order parameter depends on both the discrete spherical spectrum and the intrinsic spherical two-body binding correction.

Mean-field pairing temperature



The finite-sphere curves converge toward the flat-2D result as $1/(k_{\mathrm{F}}R)$ decreases.

Finite-temperature order parameter



At fixed $k_{\mathrm{F}}R = 5$, the order parameter decreases continuously and vanishes at the mean-field pairing temperature.

7 Comparison and Conclusions

Comparison with previous work

	He, Guo and Chien (2022)	Present treatment
Curvature parameter	Effectively $1/(k_{\mathrm{F}}R) \to 0$	Finite $1/(k_{\mathrm{F}}R)$
Spectrum	Continuum approximation	Discrete angular-momentum sums
Two-body input	Flat binding energy	Intrinsic spherical binding energy
Outcome	Flat-2D limit	Finite-size and curvature corrections

Summary

- Experimentally motivated Fermi gases on $S^2(R)$.
- Two independent parameters:

$$x = -\log(k_{\mathrm{F}} a_{2\mathrm{D}}), \qquad \frac{1}{k_{\mathrm{F}} R}.$$

- Geometry enters through:
  - the discrete spherical spectrum,
  - the intrinsic spherical binding energy.
- Flat 2D recovered as

$$\frac{1}{k_{\mathrm{F}} R} \to 0.$$

Next steps

- Pair size across the spherical BCS–BEC crossover.
- Mean-field condensate fraction and superfluid density.
- Topological and spectral-geometric observables.

Thank you