Field equation of GRT (2nd order partial differential equations for metric)

$\downarrow$ FLRW metric

two ordinary differential equations

1) $\dot{R}^2(t) + k - \frac{1}{3}\Lambda R^2(t) = \frac{8\pi G}{3c^2}\rho(t) R(t)^2$ (1)

2) $\frac{d\rho}{dt} = -3\frac{\dot{R}(t)}{R(t)}\left[\rho(t) + \frac{P(t)}{c^2}\right]$

4.1.3 Equation of State:

polytropic EoS: $P(\rho) = \underbrace{k}_{\text{parameter}}\,\rho^{\overbrace{\gamma}^{\text{polytropic exponent}}}$

Incoherent matter

$P \ll \rho c^2 \Rightarrow P = 0$

$\downarrow$

$\rho(t) R^3(t) = \text{const.}$

mass conserved

$\hat{=}$ today

Radiation dominance

$P = \frac{c^2}{3}\rho$

$\downarrow$

$\rho(t)\, R(t)^4 = \text{const}$

$\hat{=}$ right after the Big Bang

Simultaneous formal description:

$\rho_{mat}(t) R^3(t) = \text{const}$

$\rho_{rad}(t) R^4(t) = \text{const}$

$K_m = \frac{8\pi G}{3c^2}\,\rho_{mat}(t)\,R^3(t) = const$ $K_{rad} = \frac{8\pi G}{3c^2}\,\rho_{rad}(t)\,R^4(t) = const.$

$\rho(t) = \rho_{mat}(t) + \rho_{rad}(t)$ (2)

(2) in (1): Friedmann equation

$$\dot{R}^2(t) \underbrace{- \cancel{\frac{K_r}{R^2(t)}} - \frac{K_m}{R(t)} - \frac{1}{3}\Lambda R^2(t)}_{= V_{eff}(R)} = -k \qquad \left[\, \dot{} \ \hat{=}\ \frac{1}{c}\frac{d}{dt} \,\right]$$

later times

$\Rightarrow$ determines the evolution of the Universe on large

4.1.4 Special Case: $\Lambda = 0$, $K_r = 0$ $\Rightarrow$ matter-dominated Universe

$$\frac{1}{c^2}\left(\frac{dR}{dt}\right)^2 - 2G\underbrace{\frac{4\pi}{3}\rho_{mat}R^3}_{=M}\frac{1}{R} = -k \Rightarrow \underbrace{\frac{M}{2}\left(\frac{dR}{dt}\right)^2}_{\text{kinetic energy}} - \underbrace{\frac{GM^2}{R}}_{\text{gravitational energy}} = -k\,\frac{1}{2}\underbrace{Mc^2}_{\text{rest energy}}$$

$\hat{=}$ energy conservation

$\hat{=}$ gravitational collapse

$\hat{=}$ two-body Kepler problem

- $k = 1$: negative energy, bound solution ($\hat{=}$ ellipse)

  $\hat{=}$ todays observed expansion would terminate at some time, afterwards a collapse occurs

- $k = 0$: energy zero, unbound solution ($\hat{=}$ parabola)

  $\hat{=}$ expanding universe will have asymptotic expansion velocity going to zero

- $k = -1$: energy positive, unbound solution ($\hat{=}$ hyperbola)

  $\hat{=}$ expansion of Universe will never come to an end.

4.1.5 Discussion:

Cosmic scaling parameter $R(t)$: determines dynamics of Universe

- metrices
- mass distributions

Different world models correspond to different values for k, Λ, K_r, K_m as well as $R(0)$ or $\dot{R}(0)$

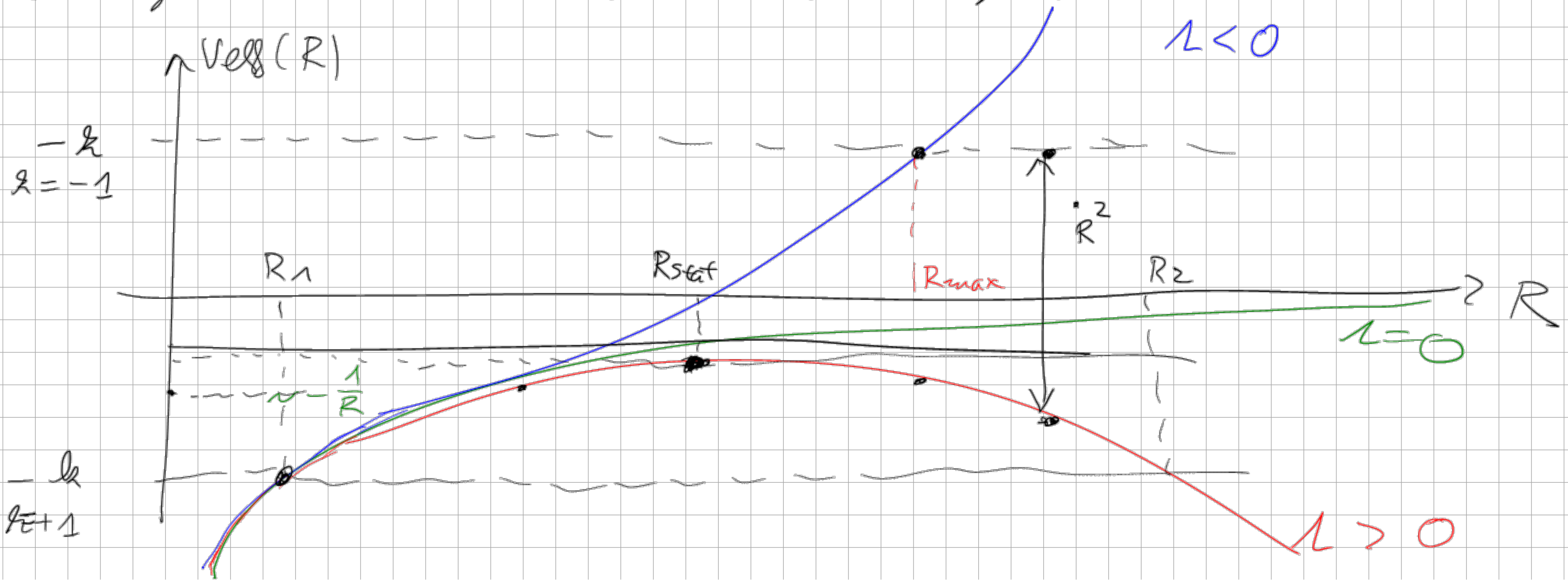
$\Rightarrow$ Qualitative discussion by looking at $V_{eff}(R)$

At first: look at $R \to 0 \;\hat{=}$ early Universe, where $1/R^2$ and $1/R$ dominate

$$\left.\begin{array}{ll} 1) \; \dot{R}^2 \approx \dfrac{K_r}{R^2} & \Rightarrow R(t) \sim t^{1/2} \\ 2) \; \dot{R}^2 \approx \dfrac{K_m}{R} & \Rightarrow R(t) \sim t^{2/3} \end{array}\right\} \quad \lim_{R \to 0} \dot{R}(t) = \infty$$

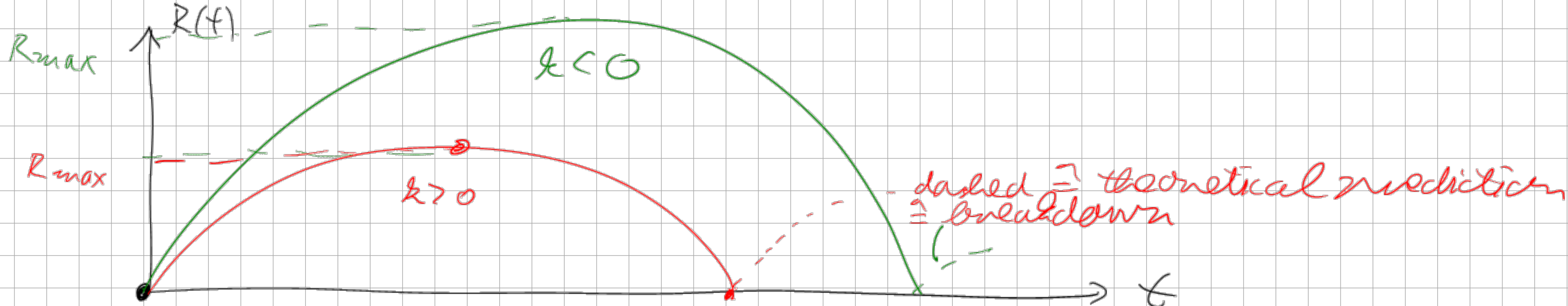
Realistic models one has $K_r \neq 0 \Rightarrow$ 1) dominates between $t = 0$ and $t = 10^{-6} t_0$ (age Universe)

For larger time 2) dominates: In the following $K_s = 0$

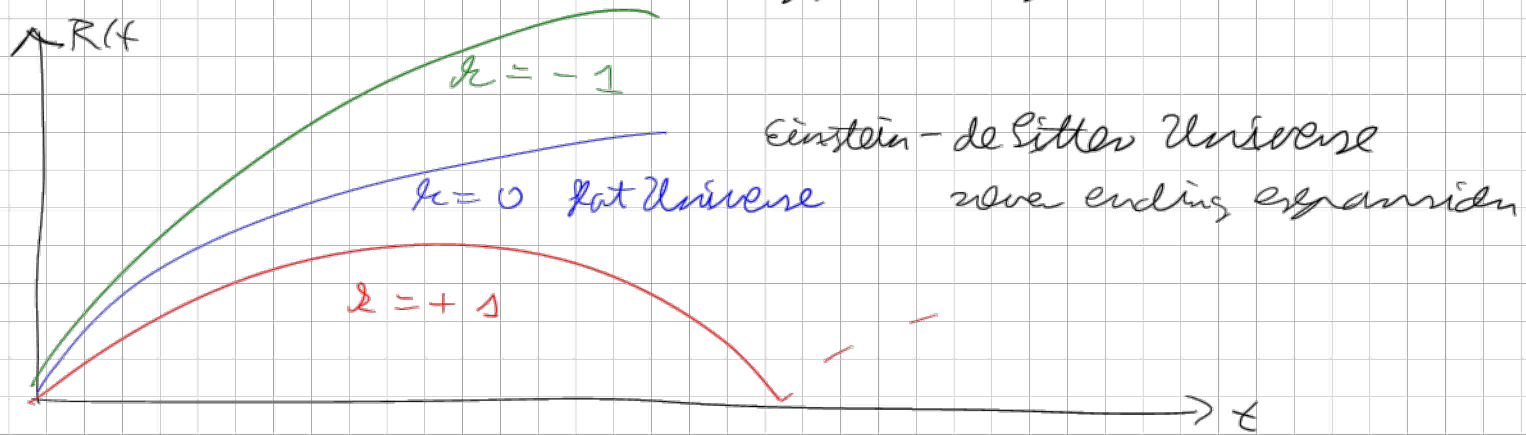


1) $\Lambda < 0$: There is always an R_{max} such that $V_{eff}(R_{max}) = -k$

irrespective of $k = 0, \pm 1$. R_{max} is the maximal value during evolution



2) $\Lambda = 0$: All discussed in Sect. 4.1.4, analogy with Kepler problem



3) $\Lambda > 0$: different cases depending upon whether $-k$ cuts $V_{eff}(R)$ never, once, twice

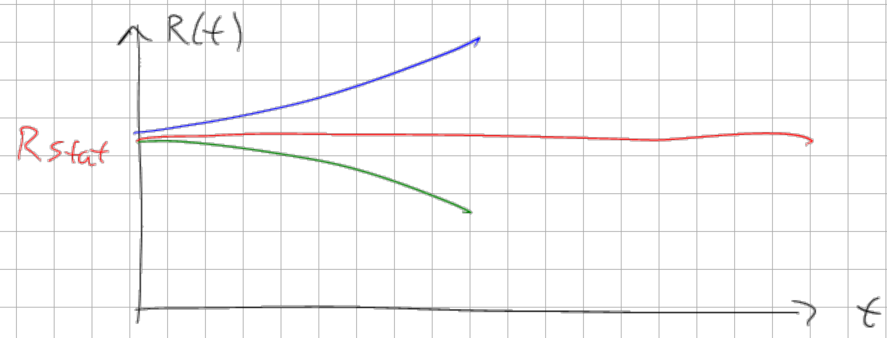
Remark:

$$V_{eff}(R) = -\frac{Km}{R} - \frac{\Lambda}{3}R^2, \quad V_{eff}'(R) = \frac{Km}{R^2} - \frac{2\Lambda}{3}R \overset{!}{=} 0 \Rightarrow R_{stat} = \sqrt[3]{\frac{3Km}{2\Lambda}}$$

$$V_{eff}(R_{stat}) = -\frac{3Km}{2R_{stat}}$$

a) At $\Lambda = \Lambda_{crit}$ and $k = +1$ the horizontal $-k = -1$ touches $V_{eff}(R)$

$$-1 = V_{eff}(R_{stat}) \Rightarrow \Lambda_{crit} = \frac{4}{9 \kappa m^2}$$



This is Einstein's static solution, but it is unstable, it is unphysical: a tiny deviation leads either to a contraction or an expansion, fate for $t \to \infty$

$$\dot{R}^2 = \frac{\Lambda}{3} R^2 \Rightarrow R(t) = R(0)\, e^{\sqrt{\frac{\Lambda}{3}}\, t}$$

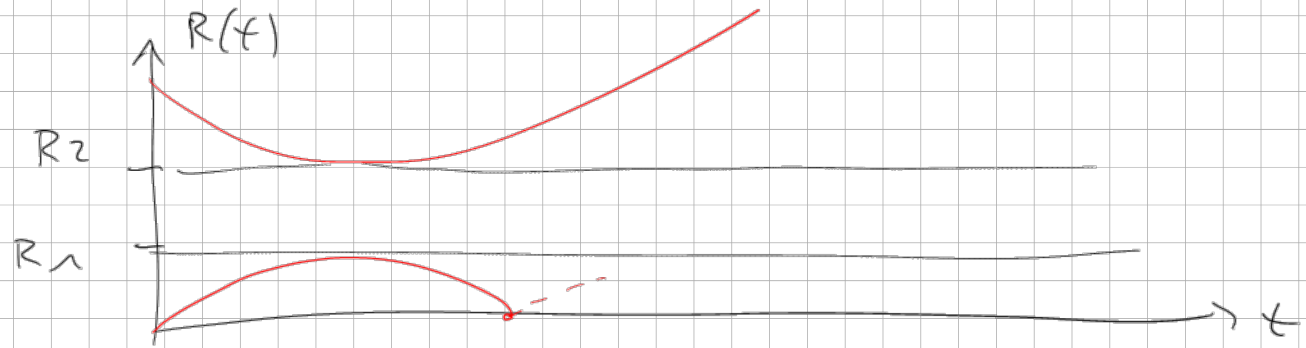
exponentially accelerated Universe

valid for all $\Lambda > 0$

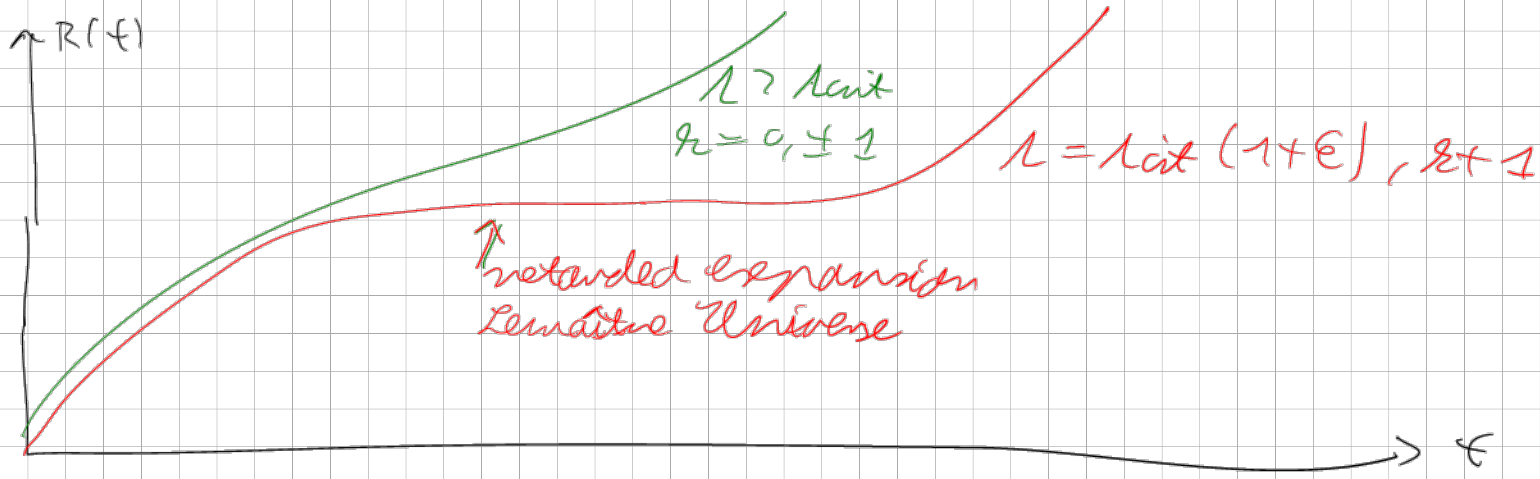
b) $\Lambda < \Lambda_{crit}$ and $k = +1$, $-k = -1$ cuts $V_{eff}(R)$ twice at R_1, R_2

$\Rightarrow R_1 < R < R_2$ forbidden

Either bound solution for $0 < R < R_1$ or unbound solution for $R > R_2$



c) $\Lambda = \Lambda_{crit}(1+\varepsilon)$ and $k = +1$ $\Rightarrow$ small $\dot{R}$



d) $\Lambda > \Lambda_{crit}$, $k = 0, \pm 1$ $\Rightarrow$ accelerated expansion

4.1.6 Dark Energy:

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} - \underbrace{\Lambda}_{\text{cosmological constant}} g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$\frac{8\pi G}{c^4} \Delta T_{00} = \frac{8\pi G}{c^4} \rho_v c^2 = \Lambda \underbrace{g_{00}}_{=1} \Rightarrow \boxed{c^2 \rho_v = \frac{\Lambda c^4}{8\pi G}}$$

(FLRW)

V_{eff}: $-\frac{Km}{R}$ and $-\frac{\Lambda}{3} R^2$ have same signs ($\Lambda > 0$)

but forces, i.e. $-\frac{\partial V_{eff}}{\partial R}$ opposite sign

$Km \mathrel{\hat{=}}$ attractive gravity $\Rightarrow$ $\Lambda > 0 \mathrel{\hat{=}}$ anti-gravity

mass term has the tendency to contract Universe, dark energy ($\Lambda > 0$) leads to expansion of Universe