Analyzing observed light from distant galaxies:

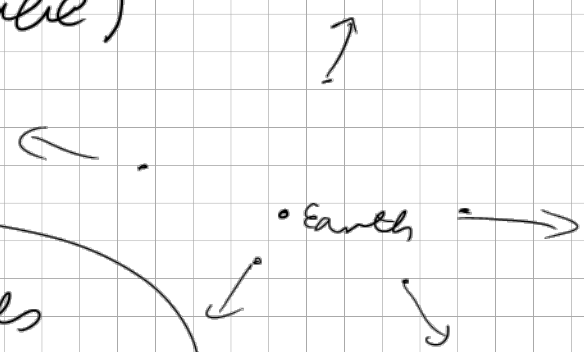
- Spectral lines of elements are shifted:

$$z = \frac{\lambda'}{\lambda} - 1 = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} - 1 \approx \frac{v}{c} \Rightarrow z > 0 \;\hat{=}\; v > 0$$ escape velocity

- Cepheides have periodic change of their brightness: comparing absolute and observed luminosity gives estimate of galaxy distance d

Hubble law: $$v = \underbrace{H_0}_{\text{Hubble constant of today}} d$$

WMAP (Wilkinson Microwave Anisotropy Probe)



space probe of NASA (2001 – 2010):

$$H_0 = 69.32 \pm 0.8 \;\frac{km}{s}\,\frac{1}{Mpc}$$

At some earlier time all these escaping galaxies were concentrated:

$$\underbrace{d = v\,t = H_0\, d}\; t \Rightarrow t_0 = \frac{1}{H_0} = \underbrace{14.02\; 10^9 a}_{\text{"age of Universe"}}$$ 4

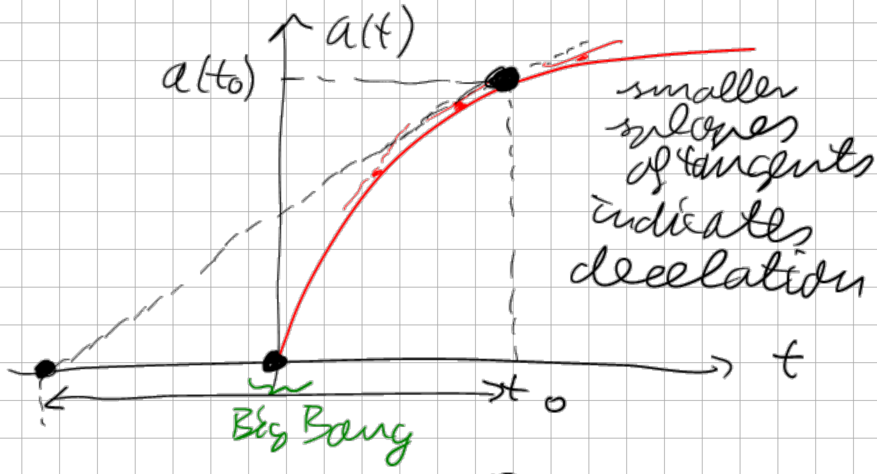
Later on: expansion of Universe

length: $d(t) \sim \underbrace{a(t)}$, velocity $\sim \dot{a}(t)$

scaling factor

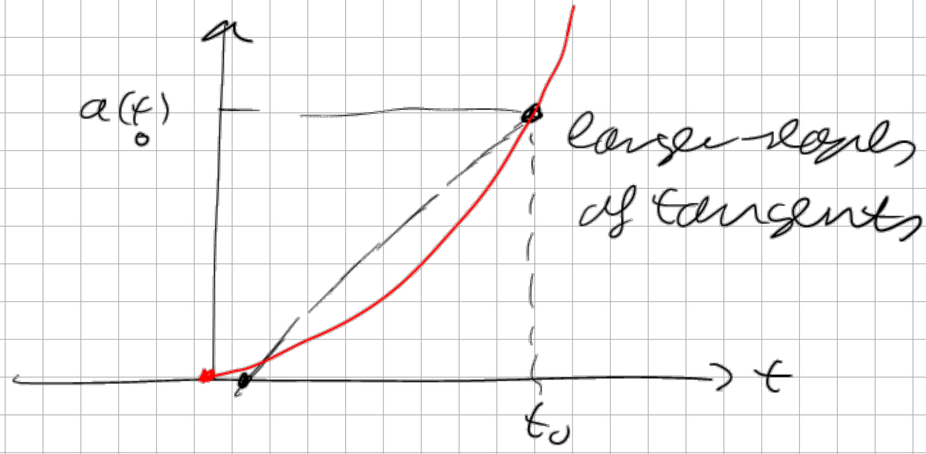
Hubble parameter: $H(t) = \frac{v}{d(t)} = \frac{\dot{a}(t)}{a(t)} \Rightarrow H_0 = H(t_0)$

decelerated expansion



Universe age is overestimated

accelerated expansion



Universe age is underestimated

Another implication of Hubble law:

$$v = H_0\, d \xRightarrow{v=c} d = \frac{c}{H_0} = 14.02 \cdot 10^9 \text{ ly}$$

Maximally possible distance of observable Universe.
This is the largest distance we get information from.
Corresponds to the horizon at Schwarzschild radius ~~rs~~ of a black hole.

2. Dynamic Star Models:

Motivation:

Robertson - Walker metric

Standard model of Cosmology:
space-time metric: isotropy and homogeneity (Cosmological principle)

Gravitational collapse of a star
- Oppenheimer - Volkoff equation
homogeneous mass distribution, Einstein field equations solved statically
- $\underbrace{R}_{\text{star radius}} > \frac{9}{8} \cdot \underbrace{r_s}_{\text{Schwarzschild radius}} = \frac{2GM}{c^2}$: stable
- $R \downarrow \frac{9}{8} r_s$
pressure at center diverges
→ gravitational collapse
→ supernova / black hole
($R < r_s$)

Advantages:
- Outside of star: Schwarzschild metric, physical properties of Schwarzschild radius
- Robertson - Walker metric allows to describe regions $< r_s$.
- Supernovae / black holes are of interest for cosmology
IA: standard candles for distance measurements.

2.1 Isotropic time-dependent metric:

General relativity: describe space-time structure by considering infinitesimal change of spacetime distance

$$ds = c \cdot \underbrace{d\tau}$$

proper time of comoving observer in gravitational field

Laboratory observer: spacetime coordinates $(x^\mu) = \begin{pmatrix} ct \\ \vec{x} \end{pmatrix}$ $\mu = 0, 1, 2, 3$

contravariant

$$c^2 d\tau^2 = ds^2 = \underbrace{g_{\mu\nu}}\, dx^\mu dx^\nu$$

covariant

space-time metric

$\hat{=}$ gravitational potentials

Absence of gravity:

• Cartesian coordinates $(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ Minkowskian metric

$$ds^2 = c^2 dt^2 - d\vec{x}^2$$

• Spherical coordinates: $\vec{x} = \begin{pmatrix} r \\ \vartheta \\ \varphi \end{pmatrix}$

$$ds^2 = c^2 dt^2 - r^2 (d\vartheta^2 + \sin^2\vartheta \, d\varphi^2) \quad \Leftarrow$$

How to describe isotropic gravitational field outside of a mass M for time-dependence:

$$ds^2 = B(r,t)\,c^2 dt^2 + \cancel{D(r,t)\,c\,dt\,dr} \underbrace{}_{-A(r,t)\,dr^2} - C(r,t)\,r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)$$

4 unknown functions A, B, C, D; two can be eliminated by transformations $\underbrace{t = f_t(r', t')}_{\text{can be chosen such that } D(r,t) = 0}$, $r = f_r(r', t')$

Schwarzschild metric

Use $r = f_r(r', t')$ such that $C(r,t) = 1$

→ now

Robertson – Walker metric

Use $r = f_r(r', t')$ such that $B(r,t) = 1$

→ later

$$\Downarrow$$

$r \to \infty$: $\lim_{r\to\infty} B(r,t) = 1 = \lim_{r\to\infty} A(r,t)$ for all t Boundary conditions

ϑ, φ: known spherical angles

t is time at infinitely distant observer

r is not the standard radius

Major difference to GRT I: time dependence

Differential geometrical consequences:

$$ds^2 = B(r,t)\,c^2 dt^2 - A(r,t)\,dr^2 - r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)$$

$$= g_{\mu\nu}\, dx^\mu\, dx^\nu$$

$$(x^\mu) = \begin{pmatrix} ct \\ r \\ \vartheta \\ \varphi \end{pmatrix} \Rightarrow (g_{\mu\nu}) = \begin{pmatrix} \underline{B(r,t)} & & & 0 \\ & -\underline{A(r,t)} & & \\ & & -r^2 & \\ 0 & & & -r^2\sin^2\vartheta \end{pmatrix}$$

$$\underbrace{g^{\mu\nu}}_{\text{contravariant metric}} \underbrace{g_{\nu\kappa}}_{\text{covariant metric}} = \delta^\mu_\kappa \Leftarrow \text{Kronecker-symbol}$$

contravariant metric $\hat{=}$ inverse of covariant metric

- Christoffel symbols: gravitational forces

$$\Gamma_{\mu\nu}{}^\lambda = \frac{1}{2} g^{\lambda\kappa}\left(+\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\kappa\mu} - \partial_\kappa g_{\mu\nu}\right)$$

- Curvature tensor:

$$R_{\mu\nu\kappa}{}^\lambda = \partial_\mu \Gamma_{\nu\kappa}{}^\lambda - \partial_\nu \Gamma_{\mu\kappa}{}^\lambda + \Gamma_{\nu\kappa}{}^\sigma \Gamma_{\mu\sigma}{}^\lambda - \Gamma_{\mu\kappa}{}^\sigma \Gamma_{\nu\sigma}{}^\lambda$$

- Ricci tensor

$$R_{\nu\kappa} = R_{\mu\nu\kappa}{}^{\mu}$$

Field equation in vacuum, i.e. outside the star: $\underline{R_{\nu\kappa} = 0}$

$$R_{00} = -\frac{B''}{2A} + \frac{B'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rA} + \frac{\ddot{A}}{2A} - \frac{\dot{A}}{4A}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) \overset{!}{=} 0$$

$$R_{11} = \frac{B''}{2A} - \frac{B'}{2B} - \frac{B'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rA} - \frac{\ddot{A}}{2B} + \frac{\dot{A}}{4B}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) \overset{!}{=} 0$$

$$R_{22} = -1 - \frac{r}{2A}\left(\frac{A'}{A} - \frac{B'}{B}\right) + \frac{1}{A} \overset{!}{=} 0$$

$$R_{33} = R_{22} \sin^2\vartheta$$

$$R_{01} = R_{10} = -\frac{\dot A}{A r} \stackrel{!}{=} 0 \Rightarrow \dot A = 0$$

$$\Rightarrow A(r,t) = A(r)$$

$R_{\mu\nu} = 0$ for all other components

$$\frac{R_{00}}{B} + \frac{R_{11}}{A} = 0 \Rightarrow -\frac{1}{rA}\left(\frac{B'}{B} + \frac{A'}{A}\right) = 0$$

$$\frac{\partial}{\partial r} \ln\left[A(r) \cdot B(r,t)\right] = 0 \Rightarrow A(r) \cdot B(r,t) = C(t)$$

$$\left.\begin{array}{l} \lim_{r\to\infty} A(r) B(r,t) = 1 = C(t) \end{array}\right\} B(r,t) = B(r) = \frac{1}{A(r)}$$

Time dependence drops out!

We get the Schwarzschild solution (of GRT I)

$$A(r) = \frac{1}{1 - \frac{r_s}{r}}, \qquad B(r) = 1 - \frac{r_s}{r}, \qquad r_s = \frac{2GM}{c^2}$$

GRT I: M is integration constant of Einstein field equations in vacuum

$\rightarrow$ non-relativistic: M is mass of gravitating object in center

GRT II: solving Oppenheimer-Volkoff equation automatically led us to that interpretation of M

Birkhoff theorem (1923): Isotropic gravitational field outside of matter distribution is static.

Analogy of Birkhoff theorem in electrodynamics:

$\operatorname{div} \vec{E} = 0$ $\qquad$ $\operatorname{div} \vec{B} = 0$

$\operatorname{rot} \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ $\qquad$ $\operatorname{rot} \vec{B} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$ $\Leftarrow$ displacement current

(first directly shown by Röntgen)

Spherical electromagnetic field:

$$\vec{E}(\vec{r},t) = E(r,t) \underbrace{\vec{e}_r}_{= \frac{\vec{r}}{r}} \quad , \quad \vec{B}(\vec{r},t) = B(r,t)\, \vec{e}_r$$

$\vec{\nabla}$ spherical coordinates $\qquad \vec{e}_r \frac{\partial}{\partial r} + \ldots$

$$\operatorname{rot} \vec{E} = \vec{e}_r \frac{\partial}{\partial r} \times E(r,t)\, \vec{e}_r = \vec{0} \quad , \quad \operatorname{rot} \vec{B} = 0$$

$$\xrightarrow{\text{Maxwell equations}} \quad \frac{\partial \vec{E}}{\partial t} = \vec{0} = \frac{\partial \vec{B}}{\partial t}$$