# 3 Relativistic Cosmology:

- Properties of FLRW metric
- Scale factor is related to spatial curvature
- Experimental observation based on distance measurements

## 3.1 Introduction:

(A1) GRT applicable on cosmological scales

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

(A2) Weyl postulate: galaxies move on time-like geodesics

$$ds^2 = c^2 \underbrace{dt^2}_{t=\tau} - \underbrace{h_{ij}\, dx^i dx^j}_{= d\sigma^2} \Rightarrow \text{geodesics: } x^i = \text{const.}$$

(A3) • Isotropy of space + cosmological principle

$\Rightarrow$ homogeneity of space $\rightarrow$ yet unknown

(1) $$d\sigma^2 = h_{ij}\, dx^i dx^j = f(r)\, dr^2 - r^2 (d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)$$

without loss of generality as always achievable via a suitable $r$-transformation

• constant curvature in space: analogy with 2D spaces

$\Rightarrow$ sphere (curvature $> 0$), plane (curvature $= 0$), hyperboloid (curvature $< 0$)

(2) $$R_{ijkl} = K(h_{il} h_{jk} - h_{ik} h_{jl}) \Rightarrow R_{jk} = 2K\, h_{jk}$$

$$R_{11} \overset{(1)}{=} \frac{f'(r)}{r\, f(r)} \overset{(2)}{=} 2k\, f(r) \qquad (3)$$

$$R_{22} \overset{(1)}{=} 1 + \frac{r f'(r)}{2 f(r)^2} - \frac{1}{f(r)} \overset{(2)}{=} 2k\, r^2 \qquad (4)$$

(3) in (4): $1 + \frac{r^2}{2 f(r)} 2k f(r) - \frac{1}{f(r)} = 2k r^2 \Rightarrow f(r) = \frac{1}{1-kr^2} \quad (5)$

(5) in (3), (4): ✓

The metric for a 3-space with constant curvature:

$$d\sigma^2 = \frac{dr^2}{1-kr^2} + r^2 (d\vartheta^2 + \sin^2\vartheta\, d\varphi^2) \qquad [k=0: \text{flat Euclidean space}]$$

Note 1: $r = \frac{\tilde{r}}{1+\frac{1}{4}k\tilde{r}^2}$

$$\Rightarrow d\sigma^2 = \underbrace{\frac{1}{(1+\frac{1}{4}k\tilde{r}^2)^2}}_{\text{conformal factor}} \cdot \Big\{ \underbrace{d\tilde{r}^2 + \tilde{r}^2 (d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)}_{\text{flat metric}} \Big\}$$

$\Rightarrow$ FLRW is conformally flat.

More precise: $g_{\mu\nu}$ must have for each time instant $t$ a spatial curvature, which is constant in space, but it can vary with time

$$ds^2 = c^2 dt^2 - \underbrace{R^2(t)}_{\text{time-dependent scaling factor}} \Big\{ \frac{dr^2}{1-kr^2} + r^2 (d\vartheta^2 + \sin^2\vartheta\, d\varphi^2) \Big\} \qquad \text{FLRW metric}$$

Note 2: $k \neq 0 \Rightarrow k = |k|\, \hat{k}, \quad \hat{k} = \pm 1$

rescaling: $\bar{r} = \sqrt{|k|}\, r$

$$ds^2 = c^2 dt^2 - \bar{R}^2(t) \left\{ \frac{d\bar{r}^2}{1-k\bar{r}^2} + \bar{r}^2 (d\vartheta^2 + \sin^2\vartheta\, d\varphi^2) \right\}, \quad \bar{R}^2 = \frac{R^2}{|k|}$$

$\Rightarrow$ valid for $k = \pm 1$ and $k = 0$

Note 3: $k = +1 \;\hat{=}$ expanding Universe, see later

$\bar{R}$ is proportional to distances between distances of two points

and " " curvature radius.

## 3.2 Geometry of 3-spaces with constant curves:

Case 1 : $k = 1$ $\quad d\sigma^2 = R\left[\frac{dr^2}{1-r^2} + r^2 (d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)\right]$

coordinate singularity

new coordinate:

$r = \sin\chi \;,\; dr = \cos\chi\, d\chi = \sqrt{1-r^2}\, d\chi$

$$\Rightarrow d\sigma^2 = R^2 \left[ d\chi^2 + \sin\chi \; (d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)\right]$$

3 D space of constant curvature embedded in 4 D Euclidean space

$$\left.\begin{aligned} w &= R\cos\vartheta \\ x &= R \sin\chi \sin\vartheta \cos\varphi \\ y &= R\sin\chi \sin\vartheta \sin\varphi \\ z &= R \sin\chi \cos\vartheta \end{aligned}\right\} \quad \begin{aligned} d\sigma^2 &= R^2 (dw^2 + dx^2 + dy^2 + dz^2) \\ w^2 + x^2 + y^2 + z^2 &= R^2 \end{aligned}$$

Case 2: $k = 0$

$$d\sigma^2 = R^2 \left[ dr^2 + r^2 (d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)\right]$$

$$d\sigma^2 = dx^2 + dy^2 + dz^2$$

$x = R r \sin\vartheta \cos\varphi$

$y = R r \sin\vartheta \sin\varphi$ $\Rightarrow$ 3D Euclidean space

$z = R r \cos\vartheta$

Case 3: $k = -1$ $\underbrace{r = \sinh\chi}_{\text{new coordinate}}$, $dr = \cosh\chi \, d\chi = \sqrt{1+r^2}\, dr$

$$\Rightarrow d\sigma^2 = R^2 \left[ d\chi^2 + \sinh^2\chi \, (d\vartheta^2 + \sin^2\vartheta + d\varphi^2 \right]$$

$$\left.\begin{aligned} w &= R \cosh\chi \\ x &= R \sinh\chi \sin\vartheta \cos\varphi \\ y &= R \sinh\chi \sin\vartheta \sin\varphi \\ z &= R \sinh\chi \cos\vartheta \end{aligned}\right\} \begin{aligned} & d\sigma^2 = R^2 \left[ -dw^2 + dx^2 + dy^2 + dz^2 \right] \\ & +w^2 - x^2 - y^2 - z^2 = R^2 \end{aligned}$$

3D hyperboloid embedded in 4D Minkowski space

Unification of 3 cases

$$\Rightarrow r = f(\chi) = \begin{cases} \sin\chi & ; \; k = +1 \quad , \; 0 \le \chi \le \pi \\ \chi & ; \; k = 0 \quad , \; 0 \le \chi < \infty \\ \sinh\chi & ; \; k = -1 \quad , \; 0 \le \chi < \infty \end{cases}$$

$$ds^2 = c^2 dt^2 - \underbrace{R^2(t) \left[ d\chi^2 + f(\chi)^2 \left( d\vartheta^2 + \sin^2\vartheta \, d\varphi^2 \right) \right]}_{= d\sigma^2 \text{ spatial distance}}$$

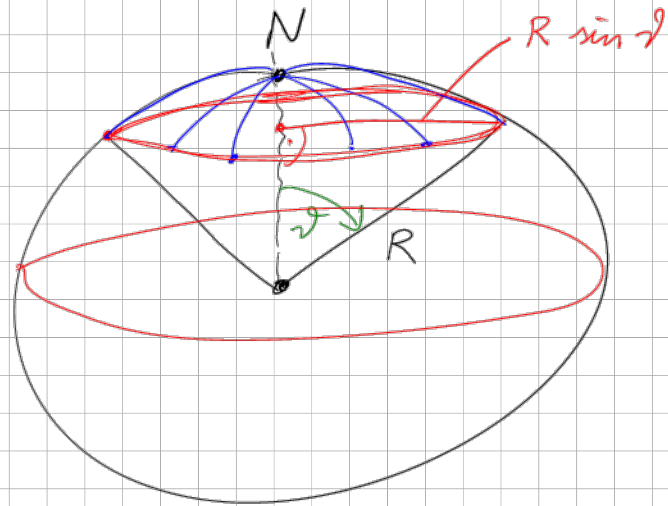
Determination of distances:

- First point identified with origin of coordinate system ($\chi = 0$, $\vartheta = 0$, $\varphi = 0$).
- Consider a path to the second point: $d\varphi = 0 = d\vartheta$. $\Rightarrow d\sigma = R(t) d\chi$

$$D = \int_0^{\chi} d\chi' \, R(t) = R(t)\,\chi \quad \Rightarrow \quad \begin{cases} \chi: \text{distance coordinate} \\ R(t): \text{scale factor changing with time} \end{cases}$$

## 3.3 Effects of Curvature:

### 3.3.1 Example of Sphere:




all points on red circle have the same distance from North Pole: $D = R\,\underset{\sim}{\vartheta}$

angle in radian measure

all red points represent a circle with circumference

$$U = 2\pi R \sin\vartheta$$

Notion:
$$\frac{U}{D} = \frac{2\pi R \sin\vartheta}{R\vartheta} = 2\pi \frac{\sin\vartheta}{\vartheta} \overset{\vartheta = \frac{D}{R}}{=} 2\pi \frac{\sin\left(\frac{D}{R}\right)}{\frac{D}{R}} \overset{R>0}{<} 2\pi$$

plane $\hat{=}$ $R \to \infty$ : $\frac{U}{D} = 2\pi$

equator: $\vartheta = \pi/2$, $\frac{U}{D} = 2\pi \cdot \frac{1}{\pi/2} = 4 < 2\pi \approx 6.3$

Now: transfer this line of reasoning from 2D to 3D

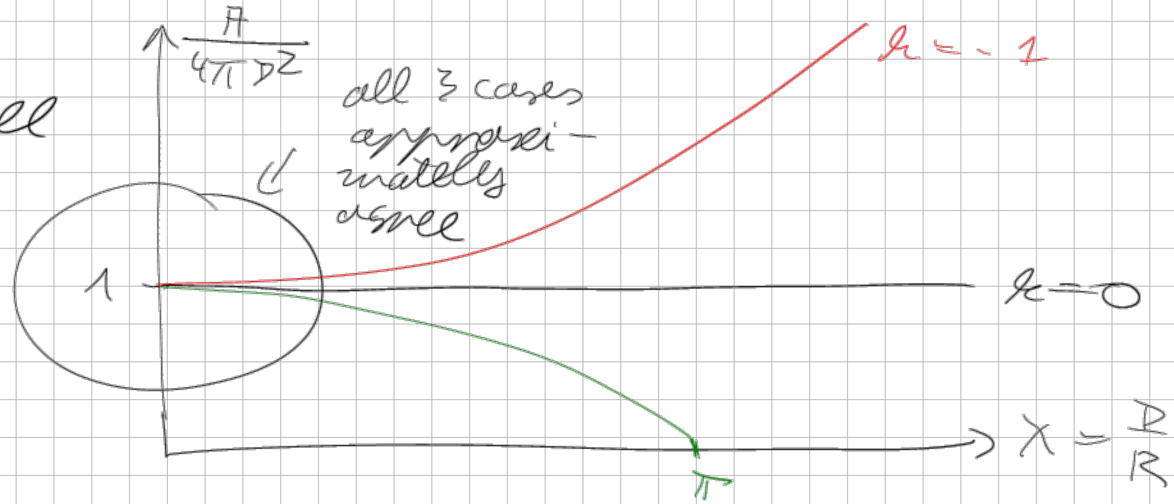
### 3.3.2 FLRW Metric:

~~First~~ point: $\chi = \vartheta = \varphi = 0$

consider all points at distance $D = R\chi$: sphere with area $A = 4\pi D^2$

$$\frac{A}{D^2} = 4\pi \frac{f(\chi)^2}{\chi^2} \;\underline{\overset{\chi = \frac{D}{R}}{=}}\; 4\pi \frac{f\left(\frac{D}{R}\right)}{\left(\frac{D}{R}\right)^2} = 4\pi \cdot \begin{cases} \sin^2\left(\frac{D}{R}\right) / \left(D/R\right)^2 \; ; & k = +1 \\ 1 \; ; & k = 0 \\ \dfrac{\sinh^2\left(\frac{D}{R}\right)}{\left(\frac{D}{R}\right)^2} \; ; & k = -1 \end{cases}$$

Moral: In order to detect curvature effects, D should not be too small




## 3.3.3 direct measurement curvature:

Assume: On cosmological scales equal distribution of galaxies with the same luminosity

Radiation power of one galaxy at D distributes at area

$$A(D) = 4\pi R^2 f\left(\frac{D}{R}\right)^2$$

Apparent luminosity:

$$\frac{1}{\ell(D)} \sim A(D) = 4\pi R^2 f\left(\frac{D}{R}\right)^2 \overset{k=0}{=} 4\pi D^2$$

$$N(D) \sim V(D) = 4\pi R^3 \int_0^{D/R} dx\, f(x)^2 \overset{k=0}{=} \frac{4\pi}{3} D^3$$

Eliminate distance D

$k=0$: $\ell \sim D^{-2}$, $D \sim \ell^{-\frac{1}{2}} \Rightarrow N \sim \ell^{-\frac{3}{2}}$

Any deviation from this power-law would indicate a non-vanishing spatial curvature. Too simplified consideration:

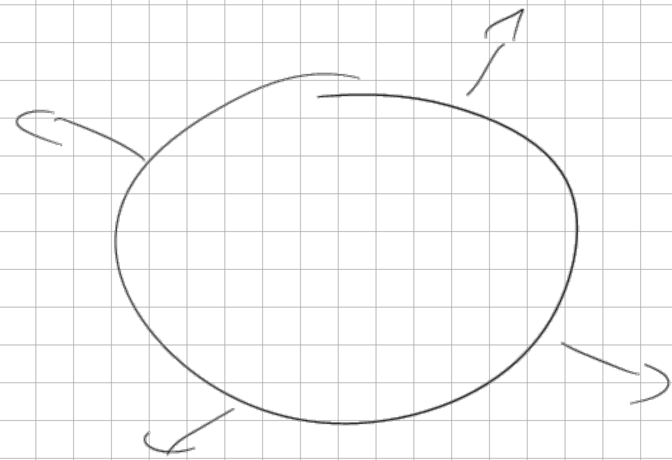
- real distribution of galaxies is more complicated
- cosmological red shift due to expansion of Universe

## 3.3.4 Finite, Unlimited Universe:

$k = 1$: volume is finite, $f(\chi) = \sin \chi$, maximally $\chi = \frac{D}{R} = \pi$ is possible

$$V = 4\pi R^3 \int_0^\pi d\chi \sin^2 \chi = 2\pi^2 R^3 \text{ is finite}$$

but unlimited like 2D balloon with expanding radius



## 3.4 Redshift – Distance Relation:

galaxy 1 — galaxy 2

emitted at $t_1$ — received at $t_2$

light trajectory along geodesics:

$d\vartheta = 0 = d\varphi$, $\chi(t_1) = 0$ to $\chi(t_2) = \chi$

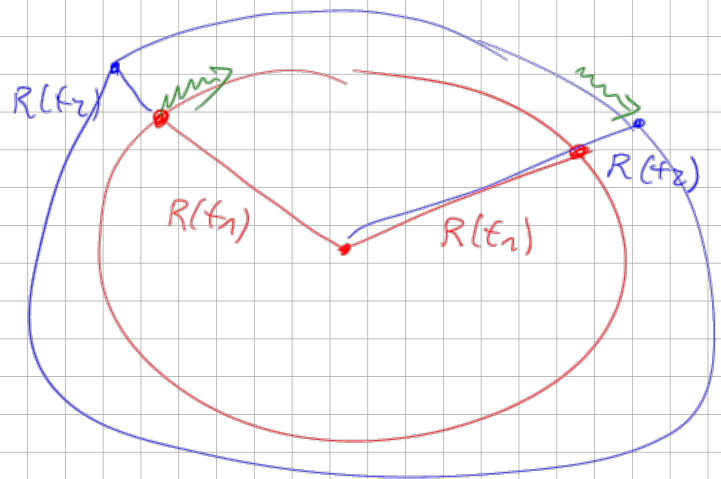
$$ds^2 = c^2 dt^2 - R^2(t) d\chi^2 = 0$$

↑ photons

$$\Rightarrow d\chi = \frac{c\,dt}{R(t)}$$

$$\chi = \int_{t_1}^{t_2} \frac{c\,dt}{R(t)} \stackrel{!}{=} \int_{t_1+\delta t_1}^{t_2+\delta t_2} \frac{c\,dt}{R(t)}$$

light maxima: $\delta t_1$ versus $\delta t_2$




$t_1 < t_2$

$\dot{R}(t_1) > 0$

$\Rightarrow \quad \delta t's \sim 10^{-14}\,s \quad \Rightarrow \quad \frac{1}{R(t)} = \text{const.}$

$\Rightarrow \quad \frac{\delta t_1}{R(t_1)} = \frac{\delta t_2}{R(t_2)} \quad \to \quad \ldots \quad \Rightarrow$ cosmological red shift