strategy for describing a gravitational collapse:

$$ds^2 = c^2 d\tau^2$$

inside of the star

isotropic time-dependent metric in Gaußian (normal) coordinates

$$= 1\cdot c^2 dt^2 - U(r,t)\, dr^2 - V(r,t)\cdot (d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)$$

radial free fall of particles

outside of the star

Schwarzschild metric

$$= \left(1-\frac{r_s}{r}\right) c^2 dt^2 - \frac{1}{1-\frac{r_s}{r}}\, dr^2 - r^2(d\vartheta^2+\sin^2\vartheta\, d\varphi^2)$$

time-independent due to Birkhoff theorem

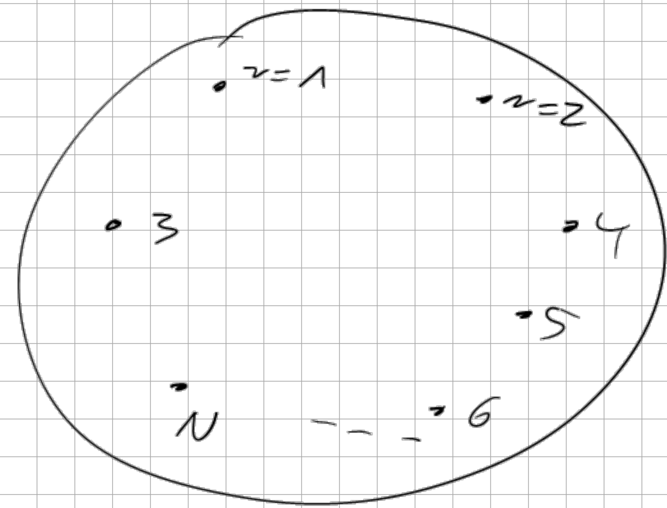
later on glue coordinates together at star surface

$$(x^\mu) = (c\underbrace{dt}_{=d\tau}, \underbrace{r, \vartheta, \varphi}_{\text{time independent}})$$

≙ comoving coordinates

star ≙ set of stones / dust particles having such a radial free fall

each dust particle $(ct_r, \underbrace{r_r, \vartheta_r, \varphi_r}_{=\text{const.}})$



• These coordinates are appropriate to describe both $r > r_s$ and $r < r_s$.

- Not all freely falling particles are described by these coordinates, only radial free fall is considered here
- Same coordinates are used for describing the evolution of the universe each dust particle corresponds then to a galaxy.

Idea: solve Einstein field equations $\Rightarrow$ Ricci tensor

Note: $x' = \frac{\partial x}{\partial r}$, $\dot{x} = \frac{\partial x}{\partial tc}$ $\qquad x = u, v$

2.4 Gravitational collapse:

Simplifying assumptions:

- Collapse should occur only isotropically
  $\rightarrow$ metric: Gaussian normal coordinates
- star = collection of stones with $(u^\mu) = (c, \underbrace{0, 0, 0}_{\text{= normal Gaussian coordinates (of star)}})$
- GRT II: equilibrium of star
  $P_{mat} = P_{grav.}$, but $R \downarrow \frac{9}{8} r_s \rightarrow P_{grav.}$ in center diverges
  collapse $\hat{=}$ nonequilibrium: $P_{mat} \ll P_{grav.}$ $\Rightarrow$ $P_{mat} \equiv 0$
- mass density is isotropic and homogeneous

$$\rho(r,t) = \begin{cases} \rho(t) & ; \quad r < r_0 \\ 0 & ; \quad r \geq r_0 \end{cases} \qquad r_0: \text{ star radius}$$

Problem: to solve the Einstein field equations

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{T}{2} g_{\mu\nu}\right)$$

$$T_{\mu\nu} = \underset{\uparrow \atop \text{fluid}}{=} \left(\rho + \frac{P}{c^2}\right) u_\mu u_\nu - P g_{\mu\nu} \overset{P=0}{=\!=} \rho(t) c^2 \delta_\mu^{\ 0} \delta_\nu^{\ 0}$$

$$(u_\mu) = (c, 0, 0, 0)$$

$$T = T_\mu^{\ \mu} = g^{\mu\nu} T_{\mu\nu} = \rho(t)\, c^2$$

$$T_{\mu\nu} - \frac{T}{2} g_{\mu\nu} = \frac{1}{2}\, \mathrm{diag}\left(\rho c^2,\ u \rho c^2,\ V\rho c^2,\ V\rho c^2 \sin^2\vartheta \right)$$

$$R_{00}: \quad \frac{\ddot{u}}{2u} + \frac{\ddot{V}}{V} - \frac{\dot{u}^2}{4u^2} - \frac{\dot{V}^2}{2V^2} = -\frac{4\pi G}{c^2}\rho \qquad (1)$$

$$R_{11}: \quad -\frac{\ddot{u}}{2} + \frac{\dot{u}^2}{4u} - \frac{\dot{u}\dot{V}}{2V} + \frac{V''}{V} - \frac{V'^2}{2V^2} - \frac{u'V'}{2uV} = -\frac{4\pi G}{c^2}\rho u \qquad (2)$$

$$R_{22}: \quad -1 - \frac{\ddot{V}}{2} - \frac{\dot{u}\dot{V}}{4u} + \frac{V''}{2u} - \frac{V'u'}{4u^2} = -\frac{4\pi G}{c^2}\rho V \qquad (3)$$

$$R_{01}: \quad \frac{\dot{V}'}{V} - \frac{\dot{V}\,V'}{2V^2} - \frac{\dot{u}\,V'}{2uV} = 0 \qquad (4)$$

(1) – (3): terms of either temporal or spatial derivatives

separation ansatz:

$$u(r,t) = R^2(t) f(r), \quad V(r,t) = S(t)^2 g(r) \qquad (5)$$

(5) in (4):

$$\frac{\dot{u}}{u} = 2\frac{\dot{R}}{R}, \quad \frac{V'}{V} = \frac{g'}{g},$$

$$\frac{\dot{V}}{V} = 2\frac{\dot{S}}{S}, \quad \frac{\dot{V}'}{V} = 2\frac{\dot{S}}{S}\frac{g'}{g}$$

$$\underbrace{2\frac{\dot{S}}{S}\cancel{\frac{g'}{g}} - \frac{1}{2}\,2\frac{\dot{S}}{S}\cancel{\frac{g'}{g}}}_{\dot{S}/S} - \frac{1}{2}\,2\frac{\dot{R}}{R}\cancel{\frac{g'}{g}} = 0 \quad \Rightarrow \quad \frac{\dot{S}}{S} = \frac{\dot{R}}{R} \quad \Rightarrow \quad S(t) = R(t)\,\underset{\text{const}}{C}$$

without loss of generality $= 1$

$\Rightarrow u(r,t) = R^2(t)\, f(r), \quad v(r,t) = R^2(t)\, g(r)$

$$ds^2 = c^2 dt^2 - R^2(t)\left\{ f(r)\, dr^2 - \cancel{R^2(t)}\, g(r)\,(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)\right\}$$

Due to general coordinate invariance: $g(r) = r^2$

$$\Rightarrow u(r,t) = R^2(t)\, f(r), \quad v(r,t) = R^2(t)\, r^2 \qquad (6)$$

Insert (6) into (2) and (3):

$$(2):\quad -\frac{f'}{r f^2} = \boxed{R\ddot{R} + 2\dot{R}^2 - \frac{4\pi G}{c^2}\rho R^2} = -2k \qquad (2')$$

$$(3):\quad -\frac{1}{r^2} + \frac{1}{r^2 f} - \frac{f'}{2 r f^2} = \boxed{R\ddot{R} + 2\dot{R}^2 - \frac{4\pi G}{c^2}\rho R^2} = -2k \qquad (3')$$

separation constants:

A) Radial degree:

$$-2k = -\frac{f'}{r f^2} \quad (7), \qquad -\frac{1}{r^2} + \frac{1}{r^2 f} - \frac{f'}{2 r f^2} = 0 \quad (8)$$

(7) in (8): pure algebraic equation for $f(r)$: $\boxed{f(r) = \frac{1}{1 - k r^2}}$ (9)

(9) fulfills (7) and (8) ✓

$$\Rightarrow ds^2 = c^2 dt^2 - R^2(t)\left[\frac{dr^2}{1 - k r^2} + r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)\right] = g_{\mu\nu}dx^\mu dx^\nu$$

This is Friedmann-Lemaître-Robertson-Walker metric for describing the temporal evolution of the universe

$R(t)$: physical dimension length; r and k are dimensionless

B) Temporal degree: $-2k = R(t)\ddot{R}(t) + 2\dot{R}^2(t) - \frac{4\pi G}{c^2}\rho(t)R(t)^2$ (10)

What determines the time dependence of $\rho(t)$?

$\Rightarrow$ Investigate energy-momentum conservation

$\hat{=}$ Bianchi identity (differential geometrical identity)

$$D_\nu T^{\mu\nu} = \partial_\nu T^{\mu\nu} + \Gamma_{\nu\lambda}{}^{\mu} T^{\lambda\nu} + \underbrace{\Gamma_{\nu\lambda}{}^{\nu}}_{=\frac{1}{\sqrt{g}}\partial_\lambda \sqrt{g}} T^{\mu\lambda}$$ $g = \det(g_{\mu\nu})$

$$= \frac{1}{\sqrt{g}}\partial_\nu\left(\sqrt{g}\,T^{\mu\nu}\right) + \Gamma_{\nu\lambda}{}^{\mu} T^{\lambda\nu} \equiv 0$$

$\mu = 0$: energy conservation, $\mu = i$: momentum conservation

$$\frac{1}{\sqrt{g}}\partial_{\cancel{\nu}\,0}\left(\sqrt{g}\,T^{0\cancel{\nu}\,0}\right) + \underbrace{\Gamma_{\cancel{\lambda\nu}\,00}{}^{0}}_{\equiv 0}\underbrace{T^{\cancel{\lambda\nu}\,00}}_{\neq 0} = 0$$

$$\Rightarrow \partial_t\left(\sqrt{g(t)}\,\rho(t)c^2\right) = 0 \longrightarrow \frac{\partial}{\partial t}\rho(t)R^3(t) = 0$$

$$g = \det(g_{\mu\nu}) = R^6(t)\frac{1}{1-kr^2}r^4\sin^2\vartheta$$

$\Rightarrow \rho(t)\cdot R^3(t) = \text{const.}$

$M_0 = \frac{4\pi}{3}\rho(t)R^3(t)$: mass of star (11)

(11) in (10): $-2k = R\ddot{R} + 2\dot{R}^2 - \frac{3M_0}{c^2 R}$ (12)

Rewrite (1) : $3\frac{\ddot{R}}{R} = -\frac{4\pi\varrho}{c^2} \overset{(11)}{=} -\frac{3GM_0}{c^2R^3}$ (13) $\Big| \cdot \frac{2}{3}R\dot{R}$

$$2\dot{R}\ddot{R} = -\frac{GM_0}{c^2}\frac{\dot{R}}{R^2}$$

$$\dot{R}^2 = \underbrace{k}_{\text{unknown}} + \frac{2GM_0}{c^2R} \quad (14) \quad \Big| \frac{M_0}{2}$$

(13) and (14) into (12) : $\boxed{k = -\varkappa}$

$$\underbrace{\frac{M_0}{2}\dot{R}^2}_{\text{kin. energ.}} - \underbrace{\frac{GM_0}{R}}_{\text{grav. energy}} = \text{const.} \quad (15)$$

This looks non-relativistically, but in fact we are in GRT:

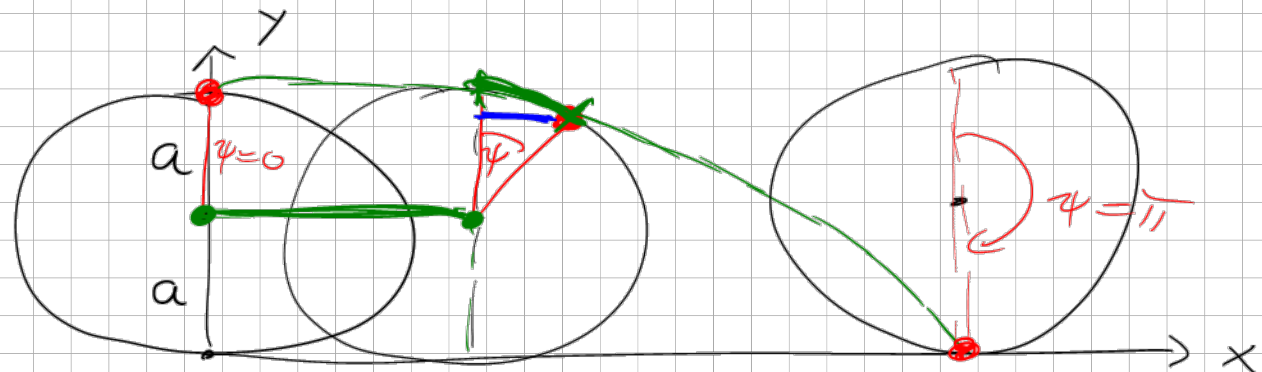
- time is proper time
- metric is not flat

initial condition: $\dot{R}(0) = 0 \Rightarrow \varkappa = \frac{2GM_0}{c^2}\frac{1}{R(0)}$

$$\Rightarrow \boxed{\dot{R}^2(t)} = \frac{2GM_0}{c^2}\left(\frac{1}{R(t)} - \frac{1}{R(0)}\right) = \boxed{\varkappa\,\frac{R(0)-R(t)}{R(t)}} \Leftarrow$$

solution: cycloid

circle rolling on x-axis without slippery



$$x = \underline{a(\psi} + \underline{\sin\psi})$$
$$y = \underline{a(1} + \cos\psi)$$

parameter representation of cycloid

$$ct(\psi) = A(\psi + \sin\psi) \qquad t(\psi=0) = 0$$

$$R(\psi) = B(1+\cos\psi) \qquad R(\psi=0) = 2B = R(0)$$

$$ct(\psi) = A(\psi + \sin\psi) \qquad ct'(\psi) = A(1+\cos\psi)$$

$$R(\psi) = \frac{R(0)}{2}(1+\cos\psi) \qquad R'(\psi) = -\frac{R(0)}{2}\sin\psi$$

$$\dot R = \frac{dR}{c\,dt} = \frac{\frac{dR}{d\psi}}{c\,\frac{dt}{d\psi}} = \frac{R'}{ct'} = -\frac{R(0)}{2c}\,\frac{\sin\psi}{A(1+\cos\psi)}$$

$$\dot R^2 = \frac{R(0)^2}{4c^2A^2}\,\frac{1-\cos^2\psi}{(1+\cos\psi)^2} = \dots = \underbrace{\frac{R(0)^2}{4A^2}}_{=k}\,\frac{R(0)-R}{R}$$

$$\cos\psi = \frac{2R}{R(0)} - 1$$

$$\Rightarrow A = \frac{R(0)}{2\sqrt{k}} \qquad \Rightarrow ct = \frac{R(0)}{2\sqrt{k}}(\psi + \sin\psi)$$

vanishing radius:

$$R(\psi) = 0 \Rightarrow \psi = \pi$$

$$t(\psi=\pi) = \boxed{\frac{\pi}{2}R(0)\sqrt{\frac{R(0)}{2GM_0}} = T}$$

this is exactly time of a radially freely falling particle starting at $R(0)$, $\dot R(0) = 0$

$$T = \frac{R(0)}{\sqrt{2GM}} \left[\arctan\sqrt{\frac{1}{z}-1} - z\sqrt{\frac{1}{z}-1} \right]_0^1$$

last lecture

2.5 Supernova:

Nova is a star whose luminosity to increase by a factor $10^2 - 10^4$ during some hours. Possible realization: white dwarf

Supernova is a star whose luminosity exceeds that of a nova by several orders of magnitude. This luminosity is larger than all the luminosities of the stars in that galaxy altogether.