13,5 Central Potential:

$\vec{A} = \vec{0}$, $\varphi(\vec{x}) = \varphi(r)$, $r = |\vec{x}|$

$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi$, $\hat{H} = c\vec{\alpha}\cdot\hat{\vec{p}} + mc^2\beta + q\varphi(r)$

1) Orbital angular momentum: $\hat{\vec{L}} = \vec{x} \times \hat{\vec{p}}$

$[\varphi(r), \hat{L}_i]_- = [\varphi(r), \varepsilon_{ijk} x_j \hat{p}_k]_- = +i\hbar\varepsilon_{ijk} x_j \underbrace{\partial_k \varphi(r)}_{= \frac{\varphi'(r)}{r} x_k}$

$= 0 \Rightarrow [\varphi(r), \hat{\vec{L}}]_- = \vec{0}$

$[\hat{H}, \hat{\vec{L}}]_- = c[\vec{\alpha}\cdot\hat{\vec{p}}, \hat{\vec{L}}]_- = -i\hbar\, \vec{\alpha} \times \hat{\vec{p}} \neq 0$

$[\alpha_e \hat{p}_e, \hat{L}_i]_- = [\alpha_e \hat{p}_e, \varepsilon_{ijk} x_j \hat{p}_k]_- = \varepsilon_{ijk}\alpha_e [\hat{p}_e, x_j]_- \hat{p}_k$

$= -i\hbar\varepsilon_{ijk}\alpha_e \delta_{je}\hat{p}_k = -i\hbar\varepsilon_{ijk}\alpha_j \hat{p}_k$

2) Spin angular momentum: $\hat{\vec{S}} = \frac{\hbar}{2}\vec{\Sigma}$, $\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$

$[\beta, \hat{\vec{S}}]_- = 0$

$[\hat{H}, \hat{\vec{S}}]_- = c[\vec{\alpha}\cdot\hat{\vec{p}}, \frac{\hbar}{2}\vec{\Sigma}]_- = \frac{c\hbar}{2}\hat{p}_e[\alpha_e, \vec{\Sigma}]_- = ic\hbar\,\vec{\alpha}\times\hat{\vec{p}} \neq 0$

$\alpha_e \Sigma_i = \begin{pmatrix} 0 & \sigma_e \\ \sigma_e & 0 \end{pmatrix}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} = \begin{pmatrix} 0 & \sigma_e\sigma_i \\ \sigma_e\sigma_i & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & \delta_{ei} + i\varepsilon_{eij}\sigma_j \\ \delta_{ei} + i\varepsilon_{eij}\sigma_j & 0 \end{pmatrix}$

$[\alpha_e, \Sigma_i]_- = 2i\varepsilon_{eij}\begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} = 2i\varepsilon_{eij}\alpha_j$

Result: $[\hat{H}, \hat{\vec{J}}]_- = 0$, $\hat{\vec{J}} = \hat{\vec{L}} + \hat{\vec{S}}$

$\Rightarrow [\hat{H}, \hat{\vec{J}}^2]_- = 0$ and $[\hat{H}, \hat{J}_z]_- = 0$.

We expect Ψ to be an eigenfunction of $\hat{\vec{J}}^2$ and $\hat{J}_z$.

spinorial spherical harmonics are Weyl spinors, which are eigenfunctions of $\hat{\vec{J}}^2$ and $\hat{J}_z$:

$$\mathcal{Y}^{j=\ell\pm\frac{1}{2},m}_{\ell,\frac{1}{2}}(\vartheta,\varphi) = \begin{pmatrix} \pm\sqrt{\frac{\ell \pm m + \frac{1}{2}}{2\ell+1}}\, Y^{m-\frac{1}{2}}_{\ell}(\vartheta,\varphi) \\ +\sqrt{\frac{\ell \mp m + \frac{1}{2}}{2\ell+1}}\, Y^{m+\frac{1}{2}}_{\ell}(\vartheta,\varphi) \end{pmatrix}$$

$\hat{\vec{J}}^2\, \mathcal{Y}^{j=\ell\pm\frac{1}{2},m}_{\ell,\frac{1}{2}}(\vartheta,\varphi) = \hbar^2 j(j+1)\, \mathcal{Y}^{j=\ell\pm\frac{1}{2},m}_{\ell,\frac{1}{2}}(\vartheta,\varphi)$

$\hat{J}_z\, \mathcal{Y}^{j=\ell\pm\frac{1}{2},m}_{\ell,\frac{1}{2}}(\vartheta,\varphi) = \hbar m\, \mathcal{Y}^{j=\ell\pm\frac{1}{2},m}_{\ell,\frac{1}{2}}(\vartheta,\varphi)$

consider we combine the two spinorial spherical harmonics to a Dirac spinor for solving the Dirac equation for a central potential?

Focus on stationary solution:

$\Psi(\vec{x},t) = \Psi e^{-\frac{i}{\hbar}Et}$, $\Psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$, $V(\vec{x}) = q\,\varphi(\vec{x}) = V(r)$

$[E - V(r)]\begin{pmatrix} \varphi \\ \chi \end{pmatrix} - mc^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \varphi \\ \chi \end{pmatrix} = c \begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix}\begin{pmatrix} \varphi \\ \chi \end{pmatrix}$

$[E - mc^2 - V(r)]\,\varphi(\vec{x}) = c\,\vec{\sigma}\cdot\vec{p}\,\chi(\vec{x})$

$[E + mc^2 - V(r)]\,\chi(\vec{x}) = c\,\vec{\sigma}\cdot\vec{p}\,\varphi(\vec{x})$

$(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B}) = \vec{A}\cdot\vec{B} + i\,\vec{\sigma}\cdot(\vec{A}\times\vec{B})$

$\vec{A} = \vec{B} = \vec{x}$: $(\vec{\sigma}\cdot\vec{x})(\vec{\sigma}\cdot\vec{x}) = \vec{x}^2 = r^2$

identity: $1 = \frac{\vec{\sigma}\cdot\vec{x}}{r}\frac{\vec{\sigma}\cdot\vec{x}}{r}$

$\vec{\sigma}\cdot\vec{p} = \frac{\vec{\sigma}\cdot\vec{x}}{r}\left[\frac{\vec{\sigma}\cdot\vec{x}}{r}\,\vec{\sigma}\cdot\vec{p}\right] = \frac{\vec{\sigma}\cdot\vec{x}}{r}\left\{\frac{\vec{x}}{r}\cdot\vec{p} + i\,\vec{\sigma}\cdot\frac{\vec{x}\times\vec{p}}{r}\right\}$

$= \vec{e}_r$; $\frac{\vec{x}\times\vec{p}}{r} = \frac{1}{r}\vec{L}$

$= (\vec{\sigma}\cdot\vec{e}_r)\left\{-i\hbar\frac{\partial}{\partial r} + i\frac{1}{\hbar r}\vec{S}\cdot\vec{L}\right\}$; $-i\hbar\frac{\partial}{\partial r}$

Let us assume that $\vec{\sigma}\cdot\vec{p}$ acts on a spinorial spherical harmonics:

$2\vec{L}\cdot\vec{S} = \vec{J}^2 - \vec{L}^2 - \vec{S}^2$

$\vec{\sigma}\cdot\vec{p} = \frac{\vec{\sigma}\cdot\vec{x}}{r}\left\{-i\hbar\frac{\partial}{\partial r} + i\frac{1}{\hbar r}\hbar^2\underbrace{[j(j+1) - l(l+1) - s(s+1)]}_{= \varkappa(j,l)}\right\}$

$j = l + \frac{1}{2} \hat{=} l = j - \frac{1}{2}$:

$\varkappa(j, j-\frac{1}{2}) = j(j+1) - (j-\frac{1}{2})(j+\frac{1}{2}) - \frac{1}{2}(\frac{1}{2}+1)$

$= \cancel{j^2} + j - \cancel{j^2} + \frac{1}{4} - \frac{3}{4} = j - \frac{1}{2} = +(\lambda - 1)$, $\lambda = j + \frac{1}{2}$

$j = l - \frac{1}{2} \hat{=} l = j + \frac{1}{2}$:

$\varkappa(j, j+\frac{1}{2}) = j(j+1) - (j+\frac{1}{2})(j+\frac{3}{2}) - \frac{3}{4}$

$= \cancel{j^2} + j - \cancel{j^2} - 2j - \frac{3}{4} - \frac{3}{4} = -j - \frac{3}{2} = -(\lambda+1)$, $\lambda = j + \frac{1}{2}$

combination of both cases:

$j = l \pm \frac{1}{2} \hat{=} l = j - \frac{1}{2}$: $\varkappa = \pm(\lambda \mp 1)$, $\lambda = j + \frac{1}{2}$

Applying $\frac{\vec{\sigma}\cdot\vec{x}}{r}$ to a spinorial spherical harmonics is much more complicated

$\vec{x} = \begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix}$, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\vec{x}\cdot\vec{\sigma} = \sin\vartheta\cos\varphi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sin\vartheta\sin\varphi \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \cos\vartheta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} \cos\vartheta & \sin\vartheta e^{-i\varphi} \\ \sin\vartheta e^{i\varphi} & -\cos\vartheta \end{pmatrix}$

It is expected that $\vec{x}\cdot\vec{\sigma}$ behaves under rotations as a scalar or a pseudoscalar. Thus, without loss of generality we can choose $\vec{x} = \vec{e}_z \Leftrightarrow \vartheta = 0$ for arbitrary φ

special choice: $\vec{x} = \vec{e}_z \Leftrightarrow \vartheta = 0$ for arbitrary φ

$\Rightarrow \vec{x}\cdot\vec{\sigma} = \vec{e}_z\cdot\vec{\sigma} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$Y_{\ell m}(\vartheta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^{(m)}(\cos\vartheta)\, e^{im\varphi}$

$Y_{\ell m}(0,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, \underbrace{P_\ell^{(m)}(1)}\, e^{im\varphi} = \sqrt{\frac{2\ell+1}{4\pi}}\,\delta_{m,0}$

$= \delta_{m,0}\, \underbrace{P_\ell(1)}$

Now we have to find the solutions for the spherical harmonics

$$\mathcal{Y}^{j=\ell\pm\frac{1}{2},m}_{\ell,\frac{1}{2}}(\vartheta,\varphi) = \frac{1}{\sqrt{2\ell+1}} \begin{pmatrix} \pm\sqrt{\ell\pm m+\frac{1}{2}}\,\delta_{m,1/2} \\ \sqrt{\ell\mp m+\frac{1}{2}}\,\delta_{m,-1/2} \end{pmatrix} \sqrt{\frac{2\ell+1}{4\pi}}$$

$$\mathcal{Y}^{j=\ell+\frac{1}{2},m}_{\ell,\frac{1}{2}}(0,\varphi) = \frac{1}{\sqrt{4\pi}} \begin{pmatrix} \sqrt{\ell+\frac{1}{2}+\frac{1}{2}}\,\delta_{m,1/2} \\ \sqrt{\ell+\frac{1}{2}+\frac{1}{2}}\,\delta_{m,-1/2} \end{pmatrix} = \sqrt{\frac{j+\frac{1}{2}}{4\pi}} \begin{pmatrix} \delta_{m,1/2} \\ \delta_{m,-1/2} \end{pmatrix}$$

$$\mathcal{Y}^{j=\ell-\frac{1}{2},m}_{\ell,\frac{1}{2}}(0,\varphi) = \frac{1}{\sqrt{4\pi}} \begin{pmatrix} -\sqrt{\ell-\frac{1}{2}+\frac{1}{2}}\,\delta_{m,1/2} \\ +\sqrt{\ell-\frac{1}{2}+\frac{1}{2}}\,\delta_{m,-1/2} \end{pmatrix} = \sqrt{\frac{j+\frac{1}{2}}{4\pi}} \begin{pmatrix} -\delta_{m,1/2} \\ \delta_{m,-1/2} \end{pmatrix}$$

summarizing: $\mathcal{Y}^{j,m}_{\ell=j\pm\frac{1}{2},\frac{1}{2}}(0,\varphi) = \sqrt{\frac{j+\frac{1}{2}}{4\pi}} \begin{pmatrix} \pm\delta_{m,1/2} \\ \delta_{m,-1/2} \end{pmatrix}$

$$\vec{j}\cdot\vec{e}_z\, \mathcal{Y}^{j,m}_{\ell=j\mp\frac{1}{2},\frac{1}{2}}(0,\varphi) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \sqrt{\frac{j+\frac{1}{2}}{4\pi}} \begin{pmatrix} \pm\delta_{m,1/2} \\ \delta_{m,-1/2} \end{pmatrix}$$

$$= -\sqrt{\frac{j+\frac{1}{2}}{4\pi}} \begin{pmatrix} \mp\delta_{m,1/2} \\ \delta_{m,-1/2} \end{pmatrix} = -\,\mathcal{Y}^{j,m}_{\ell=j\pm\frac{1}{2},\frac{1}{2}}(0,\varphi)$$

due to independence of ϑ,φ:

$$\vec{j}\cdot\frac{\vec{x}}{r}\, \mathcal{Y}^{j,m}_{\ell=j\mp\frac{1}{2},\frac{1}{2}}(\vartheta,\varphi) = -\,\mathcal{Y}^{j,m}_{\ell=j\pm\frac{1}{2},\frac{1}{2}}(\vartheta,\varphi)$$

↑ pseudo-scalar

sign exchange

Thus we have to consider always to combine two spherical harmonics to a Dirac spinor:

$$\Psi_A(\vec{x}) = \begin{pmatrix} u_A(\vec{x}) \\ v_A(\vec{x}) \end{pmatrix} = \begin{pmatrix} u_A(r) & \mathcal{Y}^{j,m}_{j-\frac{1}{2},\frac{1}{2}}(\vartheta,\varphi) \\ (-i)\, v_A(r) & \mathcal{Y}^{j,m}_{j+\frac{1}{2},\frac{1}{2}}(\vartheta,\varphi) \end{pmatrix}$$

convenient convention

$$\Psi_B(\vec{x}) = \begin{pmatrix} u_B(\vec{x}) \\ v_B(\vec{x}) \end{pmatrix} = \begin{pmatrix} u_B(r) & \mathcal{Y}^{j,m}_{j+\frac{1}{2},\frac{1}{2}}(\vartheta,\varphi) \\ -i\, v_B(r) & \mathcal{Y}^{j,m}_{j-\frac{1}{2},\frac{1}{2}}(\vartheta,\varphi) \end{pmatrix}$$

Option A:

1) $$[E - Mc^2 - V(r)]\, u_A(r)\, \mathcal{Y}^{j,m}_{j-\frac{1}{2},\frac{1}{2}}(\vartheta,\varphi)$$

$$= -c\, \frac{\vec{\sigma}\cdot\vec{x}}{r}\, \mathcal{Y}^{j,m}_{j+\frac{1}{2},\frac{1}{2}}(\vartheta,\varphi) \left\{ -i\hbar \frac{\partial}{\partial r} + i\frac{\hbar}{r}(-\lambda - 1) \right\} (-i)\, v_A(r)$$

$$[E - Mc^2 - V(r)]\, u_A(r) = \underbrace{-c(-i\hbar)(-i)}_{= \hbar c} \left\{ \frac{\partial}{\partial r} + \frac{\lambda+1}{r} \right\} v_A(r)$$

2) $$[E + Mc^2 - V(r)]\, (-i)\, v_A(r)\, \mathcal{Y}^{j,m}_{j+\frac{1}{2},\frac{1}{2}}(\vartheta,\varphi)$$

$$= -c\, \frac{\vec{\sigma}\cdot\vec{x}}{r}\, \mathcal{Y}^{j,m}_{j+\frac{1}{2},\frac{1}{2}}(\vartheta,\varphi) \left\{ -i\hbar \frac{\partial}{\partial r} + i\frac{\hbar}{r}(\lambda - 1) \right\} u_A(r)$$

$$[E + Mc^2 - V(r)]\, v_A(r) = \underbrace{-i(-i\hbar c)}_{= -\hbar c} \left\{ \frac{\partial}{\partial r} - \frac{\lambda - 1}{r} \right\} u_A(r)$$

Option B: is treated in a similar way

$$[E - Mc^2 - V(r)]\, u(r) - \hbar c \left(\frac{\partial}{\partial r} + \frac{\lambda+1}{r} \right) v(r) = 0$$

$$[E + Mc^2 - V(r)]\, v(r) + \hbar c \left(\frac{\partial}{\partial r} - \frac{\lambda-1}{r} \right) u(r) = 0$$

two coupled first-order differential equations. Similar to Schrödinger theory we introduce the substitution

$$u(r) = \frac{F(r)}{r}, \quad u'(r) = \frac{F'(r)}{r} - \frac{1}{r^2} F(r)$$

$$v(r) = \frac{G(r)}{r}, \quad v'(r) = \frac{G'(r)}{r} - \frac{1}{r^2} F(r)$$

$$G'(r) + \frac{\lambda}{r} G(r) = \left(\frac{E - Mc^2}{\hbar c} - \frac{V(r)}{\hbar c} \right) F(r)$$

$$F'(r) + \frac{\lambda}{r} F(r) = \left(\frac{E + Mc^2}{\hbar c} - \frac{V(r)}{\hbar c} \right) G(r)$$

Bound states are normalizable, i.e.

$$\int_0^\infty dr\, r^2 \left\{ |u(r)|^2 + |v(r)|^2 \right\} = 1$$

$V(r) = -\frac{\alpha \hbar c}{r} \Rightarrow \frac{V(r)}{\hbar c} = -\frac{\alpha}{r}$

$G'(r) + \frac{\lambda}{r} G(r) = \left(\frac{E - mc^2}{\hbar c} + \frac{\alpha}{r}\right) F(r)$

$-F'(r) + \frac{\lambda}{r} F(r) = \left(\frac{E + mc^2}{\hbar c} + \frac{\alpha}{r}\right) G(r)$

Aim: bound states for $0 < E < mc^2 \Rightarrow E - mc^2 < 0$

Asymptotic behaviour in limit $r \to \infty$

$$G'(r) = -\frac{mc^2 - E}{\hbar c} F(r), \quad F'(r) = -\frac{mc^2 + E}{\hbar c} G(r)$$

$$F''(r) = -\frac{mc^2 + E}{\hbar c} G'(r) = +\frac{(mc^2 - E)(mc^2 + E)}{\hbar^2 c^2} F(r)$$

$F''(r) = e^{-\beta r}$, $r \to \infty \Rightarrow \beta = (\pm) \frac{\sqrt{m^2c^4 - E^2}}{\hbar c}$ → normalization

Asymptotic behaviour near origin, i.e. limit $r \to 0$

$$G'(r) + \frac{\lambda}{r} G(r) = \frac{\alpha}{r} F(r)$$

$$-F'(r) + \frac{\lambda}{r} F(r) = \frac{\alpha}{r} G(r)$$

Solution ansatz: $F(r) = a_0 r^\gamma$, $G(r) = b_0 r^\gamma$

$$\gamma b_0 r^{\gamma-1} + \lambda b_0 r^{\gamma-1} = \alpha a_0 r^{\gamma-1} \qquad \alpha a_0 - (\gamma + \lambda) b_0 = 0$$

$$-\gamma a_0 r^{\gamma-1} + \lambda a_0 r^{\gamma-1} = \alpha b_0 r^{\gamma-1} \Rightarrow -(\gamma - \lambda) a_0 - \alpha b_0 = 0$$

$$\begin{pmatrix} \alpha & -(\gamma+\lambda) \\ -(\gamma-\lambda) & -\alpha \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{vmatrix} \alpha & -(\gamma+\lambda) \\ -(\gamma-\lambda) & -\alpha \end{vmatrix} = 0$$

$a_0, b_0 \neq 0$

$$-\alpha^2 - (\gamma + \lambda)(\gamma - \lambda) = -\alpha^2 - \gamma^2 + \lambda^2 = 0 \Rightarrow \gamma = (\pm)\sqrt{\lambda^2 - \alpha^2}$$

normalization

$$\gamma = \sqrt{(j + \tfrac{1}{2})^2 - \alpha^2}, \quad \alpha \approx \frac{1}{137}$$

Note: nucleus of charge Ze corresponds to substitution

$\alpha \to \alpha Z$

$\gamma = \sqrt{(j + \frac{1}{2})^2 - Z^2\alpha^2} \Rightarrow Z \cdot \alpha < 1$

Interpolation between $r \to 0$ and $r \to \infty$ behaviour

$F(r) = r^\delta e^{-\beta r} f(r)$

$F'(r) = \left[\left(\frac{\delta}{r} - \beta\right) f(r) + f'(r)\right] r^\delta e^{-\beta r}$

$G(r) = r^\gamma e^{-\beta r} g(r)$

$G'(r) = \left[\left(\frac{\gamma}{r} - \beta\right) g(r) + g'(r)\right] r^\gamma e^{-\beta r}$

$$\left(\frac{\gamma+\lambda}{r} - g\right) g(r) + g'(r) = \left(\frac{E-mc^2}{\hbar c} + \frac{\alpha}{r}\right) f(r)$$

$$\left(\frac{-\gamma+\lambda}{r} + g\right) f(r) - f'(r) = \left(\frac{E+mc^2}{\hbar c} + \frac{\alpha}{r}\right) g(r)$$

Taylor series:

$$g(r) = \sum_{n=0}^{\infty} a_n r^n \quad , \quad f(r) = \sum_{n=0}^{\infty} b_n r^n$$

convert first differential equation into recursion relation

$$(\gamma+\lambda) \sum_{n=0}^{\infty} b_n r^{n-1} + \sum_{n=1}^{\infty} n\, b_n r^{n-1} - g \sum_{n'=1}^{\infty} b_{n'-1} r^{n'} = \frac{E-mc^2}{\hbar c} \sum_{n'=1}^{\infty} a_{n'-1} r^{n'} + \alpha \sum_{n=0}^{\infty} a_n r^{n+1} \quad | \cdot r$$

$n=0$: $\alpha a_0 = (\gamma+\lambda) b_0$ $\Leftarrow$ valid due to choice of γ

$$(n+\gamma+\lambda) b_n - g\, b_{n-1} = \frac{E-mc^2}{\hbar c} a_{n-1} + \alpha a_n$$

similar procedure for second differential equation:

$$(-n-\gamma+\lambda) a_n + g\, a_{n-1} = \frac{E+mc^2}{\hbar c} b_{n-1} + \alpha b_n$$

$$\begin{pmatrix} \alpha & -(n+\gamma+\lambda) \\ -n-\gamma+\lambda & -\alpha \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} \frac{mc^2-E}{\hbar c} & -g \\ -g & \frac{mc^2+E}{\hbar c} \end{pmatrix} \begin{pmatrix} a_{n-1} \\ b_{n-1} \end{pmatrix} \quad (*)$$

singular matrix due to vanishing determinant

There exists a left eigenvector of this singular matrix at eigenvalue 0

$$\left(\frac{mc^2+E}{\hbar c}, +g\right) \begin{pmatrix} \frac{mc^2-E}{\hbar c} & -g \\ -g & \frac{mc^2+E}{\hbar c} \end{pmatrix} = 0$$

Multiply (*) with this left eigenvector:

$$\left(\frac{mc^2+E}{\hbar c}, g\right) \begin{pmatrix} \alpha & -n-\gamma+\lambda \\ -n-\gamma+\lambda & -\alpha \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix} = 0$$

Relation between Taylor coefficients a_n and b_n:

$$\frac{mc^2+E}{\hbar c}\left[\alpha a_n - (n+\gamma+\lambda) b_n\right] + g\left[(-n-\gamma+\lambda) a_n - \alpha b_n\right] = 0$$

$$\left[\frac{mc^2+E}{\hbar c}\alpha + g(-n-\gamma+\lambda)\right] a_n - \left[\frac{mc^2+E}{\hbar c}(n+\gamma+\lambda) + \alpha g\right] b_n = 0$$

One can show that Taylor series $g(r)$, $f(r)$ exponentially diverge: they must be truncated

$$\frac{mc^2-E}{\hbar c}\, a_{\underbrace{n-1}_{=n_r}} - \varrho\, b_{\underbrace{n-1}_{=n_r}} = 0 \qquad \text{radial quantum number}$$

$$\begin{pmatrix} \frac{mc^2-E}{\hbar c} & -\varrho \\ \frac{mc^2+E}{\hbar c}\alpha + \varrho\left(-\frac{n_r}{\gamma}-\gamma+\lambda\right) & -\left[\frac{mc^2+E}{\hbar c}\left(\frac{n_r}{\gamma}+\gamma+\lambda\right)+\lambda\varrho\right] \end{pmatrix} \begin{pmatrix} a_{n_r} \\ b_{n_r} \end{pmatrix} = 0$$

non-vanishing solution:

$$-\varrho^2(n_r+\gamma+\lambda) + \frac{mc^2-E}{\hbar c}\alpha\varrho + \varrho\frac{mc^2+E}{\hbar c}\alpha + \varrho^2(-n_r-\gamma+\lambda) = 0$$

$$-\varrho^2(n_r+\gamma) + \frac{E}{\hbar c}\alpha\varrho = 0 \qquad \text{KORREKT vom Zuvorher}$$

$$\varrho^2(n_r+\gamma)^2 = \frac{(mc^2)^2-E^2}{\hbar^2c^2}(n_r+\gamma)^2 = \frac{E^2}{\hbar^2c^2}\alpha^2$$

$$m^2c^4\cdot(n_r+\gamma)^2 = E^2\left\{\alpha^2+(n_r+\gamma)^2\right\}$$

$$E^2 = \frac{m^2c^4(n_r+\gamma)^2}{\alpha^2+(n_r+\gamma)^2} = \frac{m^2c^4}{1+\frac{\alpha^2}{(n_r+\gamma)^2}}$$

$$E = \frac{mc^2}{\sqrt{1+\frac{\alpha^2}{\left[n_r+\sqrt{(j+\frac12)^2-\alpha^2}\right]^2}}}$$

$n_r + j + \frac12 = n$ principal quantum number

$$E_{nj} = \frac{mc^2}{\sqrt{1+\frac{\alpha^2}{\left[n-j-\frac12+\sqrt{(j+\frac12)^2-\alpha^2}\right]^2}}}$$

$$\sqrt{(j+\tfrac12)^2-\alpha^2} = j+\tfrac12-\frac{\alpha^2}{2}\frac{1}{j+\frac12}$$

$$n-j-\tfrac12+j+\tfrac12-\frac{\alpha^2}{2}\frac{1}{j+\frac12} = n-\frac{\alpha^2}{2}\frac{1}{j+\frac12}$$

$$= n^2 - \frac{\alpha^2 n}{j+\frac12}+\ldots = n^2\left\{1-\frac{\alpha^2}{n(j+\frac12)}\right\}$$

$$\simeq mc^2\left\{1+\frac{\alpha^2}{n^2}\left[1+\frac{\alpha^2}{n(j+\frac12)}+\ldots\right]^{-\frac12}\right\} \qquad \text{2. Ordnung Taylor-Reihe}$$

$$= mc^2\left\{1-\frac12\frac{\alpha^2}{n^2}\left(1+\frac{\alpha^2}{n(j+\frac12)}+\ldots\right)+\boxed{\frac{-1}{2}\left(\frac{-3}{2}\right)\frac{\alpha^4}{n^4}}+\ldots\right\}$$

$$= mc^2 + \frac{-1}{2}mc^2\frac{\alpha^2}{n^2} + \frac{-1}{2}mc^2\frac{\alpha^2}{n^2}\cdot\frac{\alpha^2}{n}\left[\frac{1}{j+\frac12}-\frac{3}{4n}\right]+\ldots \checkmark$$