# Klein-Gordon-Theory:

Schrödinger theory: How to set up a quantum field theory?

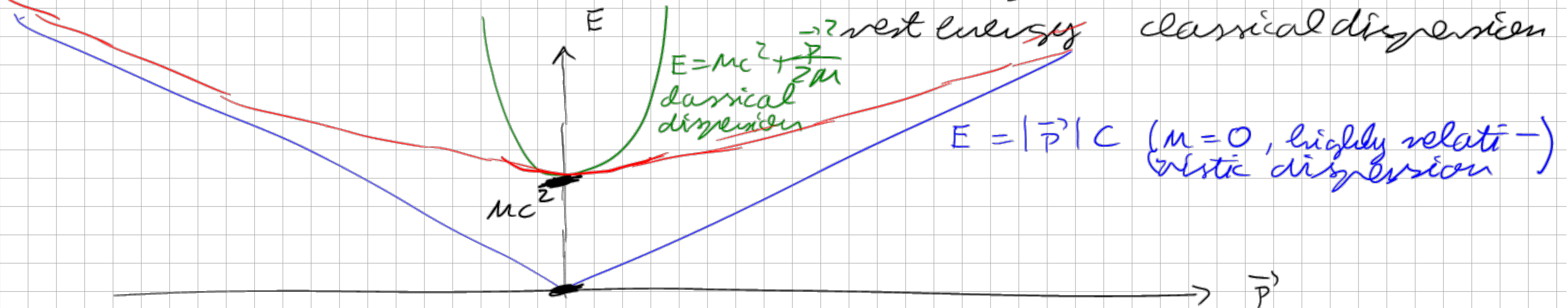
- dispersion: $E = \vec{p}^{\,2}/2m$
- Jordan rule: $\vec{p} \to \frac{\hbar}{i}\vec{\nabla}$, $E \to i\hbar\frac{\partial}{\partial t}$
- minimal coupling

## 12.2 Derivation of Klein-Gordon equation:

relativistic dispersion: $E = \sqrt{\vec{p}^{\,2}c^2 + m^2c^4} \Rightarrow \frac{E^2}{m^2c^4} - \frac{\vec{p}^{\,2}c^2}{m^2c^4} = 1$

non-relativistic limit: $|\vec{p}| \ll mc$

$$E = mc^2\sqrt{1 + \frac{\vec{p}^{\,2}}{m^2c^2}} = mc^2\left\{1 + \frac{1}{2}\frac{\vec{p}^{\,2}}{2m^2c^2} + \dots\right\} = \underbrace{mc^2}_{\text{rest energy}} + \underbrace{\frac{\vec{p}^{\,2}}{2m}}_{\text{classical dispersion}} + \dots$$




$$E = \sqrt{\vec{p}^{\,2}c^2 + m^2c^4} \Rightarrow E^2 = \vec{p}^{\,2}c^2 + m^2c^4$$

Jordan rule

$$\underbrace{\left(i\hbar\frac{\partial}{\partial t}\right)}_{=\hat{E}}\Psi(\vec{x},t) = \left\{\underbrace{\left(\frac{\hbar}{i}\vec{\nabla}\right)}_{\hat{\vec{p}}^{\,2}}{}^{2}c^2 + m^2c^4\right\}\Psi(\vec{x},t) \quad \Big| :(\hbar^2c^2) \qquad (1)$$

(complex) Klein-Gordon field

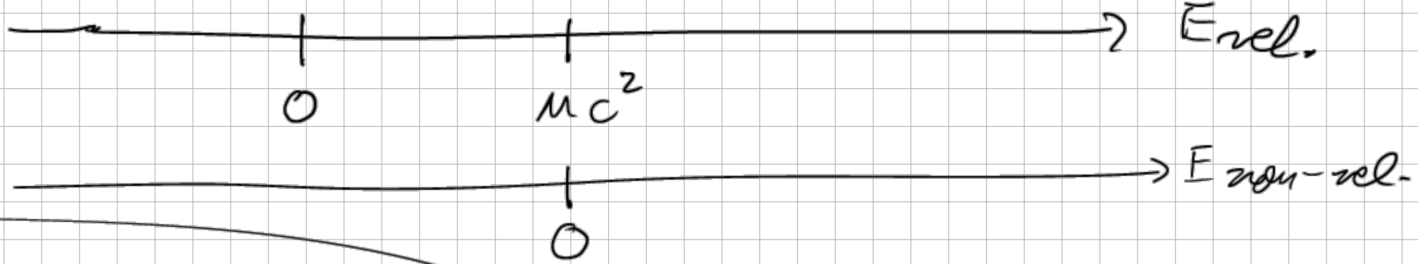
$$\underbrace{\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right.}_{= \Box \text{ "quabla operator"}} \left. + \boxed{\left(\frac{mc}{\hbar}\right)^2}\right)\Psi(\vec{x},t) = 0$$

"quabla operator" $\longrightarrow$ wave equation

"mass" term $\Rightarrow$ Compton wave length

$$\lambda_c = \frac{h}{mc} = 2\pi\frac{\hbar}{mc}\,; \qquad \pi^{\pm}: \; mc^2 = 140\,\text{MeV} \Rightarrow \lambda_c = 9\text{ fm}$$

Non-relativistic limit of Klein-Gordon equation




$$E_{rel} = mc^2 + E_{non\text{-}rel.}$$

$$\Psi(\vec{x},t) \overset{!}{=} \psi(x,t)\, e^{-\frac{i}{\hbar}mc^2 t} \qquad (2)$$

Klein-Gordon field $\quad$ Schrödinger field

$$\underbrace{i\hbar\frac{\partial}{\partial t}}_{\hat{E}}\Psi = \left(\underbrace{i\hbar\frac{-i}{\hbar}mc^2}_{mc^2} + \underbrace{i\hbar\frac{\partial}{\partial t}}_{\hat{E}}\right)\psi\, e^{-\frac{i}{\hbar}mc^2 t}$$

(2) in (1):

$$\left[\frac{1}{c^2}\left\{\cancel{\frac{\partial^2\psi}{\partial t^2}} - 2\frac{i}{\hbar}mc^2\frac{\partial\psi}{\partial t} - \cancel{\frac{m^2c^4}{\hbar^2}\psi}\right\} - \Delta\psi + \cancel{\frac{m^2c^2}{\hbar^2}\psi}\right] e^{-\frac{i}{\hbar}mc^2 t} = 0$$

formal limit: $c \to \infty$ $\Rightarrow$ $i\hbar \frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\Delta\psi$ ✓

## 12.3. Continuity equation for Klein-Gordon equation:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta + \frac{m^2c^2}{\hbar^2}\right)\Psi(\vec{x},t) = 0 \quad \Big| \; \Psi^*(\vec{x},t)\cdot$$

$$-\quad\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta + \frac{m^2c^2}{\hbar^2}\right)\Psi^*(\vec{x},t) = 0 \quad \Big| \cdot \Psi(\vec{x},t)\cdot$$

$$\underbrace{\frac{1}{c^2}\left\{\Psi^*\frac{\partial^2\Psi}{\partial t^2} - \Psi\frac{\partial^2\Psi^*}{\partial t^2}\right\}}_{\frac{\partial}{\partial t}\left\{\Psi^*\frac{\partial\Psi}{\partial t} - \Psi\frac{\partial\Psi^*}{\partial t}\right\}} + \underbrace{\Psi\Delta\Psi^* - \Psi^*\Delta\Psi}_{\vec{\nabla}\left\{\Psi\vec{\nabla}\Psi^* - \Psi^*\vec{\nabla}\Psi\right\}} = 0 \quad \Big| \cdot$$

multiply by some quantity

$$\Rightarrow \quad \frac{\partial}{\partial t}\rho + \operatorname{div}\vec{j} = 0 \qquad \text{continuity equation}$$

$$\rho = \frac{\kappa}{c^2}\left\{\Psi^*\frac{\partial\Psi}{\partial t} - \Psi\frac{\partial\Psi^*}{\partial t}\right\}, \qquad \vec{j} = \kappa\left\{\Psi\vec{\nabla}\Psi^* - \Psi^*\vec{\nabla}\Psi\right\}$$

$\kappa$ is unknown $\Rightarrow$ determine $\kappa$ from non-relativistic limit

$$\Psi = \psi\, e^{-\frac{i}{\hbar}mc^2t} \qquad \Psi^* = \psi^*\, e^{+\frac{i}{\hbar}mc^2t}$$

$$\rho = \frac{\kappa}{c^2}\Big\{\psi^*\cancel{e^{\frac{i}{\hbar}mc^2t}}\Big[\frac{\partial\psi}{\partial t} - \underline{\frac{i}{\hbar}mc^2\psi}\Big]\cancel{e^{-\frac{i}{\hbar}mc^2t}}$$

$$-\psi\,\cancel{e^{-\frac{i}{\hbar}mc^2t}}\Big[\frac{\partial\psi^*}{\partial t} + \underline{\frac{i}{\hbar}mc^2\psi^*}\Big]\cancel{e^{\frac{i}{\hbar}mc^2t}}\Big\}$$

$$= \frac{\kappa}{\cancel{c^2}}\frac{-2i}{\hbar}m\cancel{c^2}\,\psi^*\psi + \cancel{\frac{\kappa}{c^2}\left\{\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right\}}$$

$\kappa$ does not contain $c$: $\lim_{c\to\infty}$

$\Rightarrow \quad \rho \stackrel{!}{=} \psi^* \psi = |\psi|^2 \quad \Rightarrow \quad K = -\frac{\hbar}{2im}$

$$\underbrace{\rho = \frac{i\hbar}{2mc^2}\left\{\Psi^* \frac{\partial \Psi}{\partial t} - \Psi \frac{\partial \Psi^*}{\partial t}\right\}}_{\text{very different from Schrödinger theory}}, \quad \underbrace{\vec{j} = \frac{\hbar}{2im}\left\{\Psi^* \vec{\nabla}\Psi - \Psi \vec{\nabla}\Psi^*\right\}}_{\text{the same as in Schrödinger theory}}$$

How to interpret this physically?

$$\Psi(\vec{x},t) = \psi(\vec{x})\, e^{\mp \frac{i}{\hbar} mc^2 t}$$

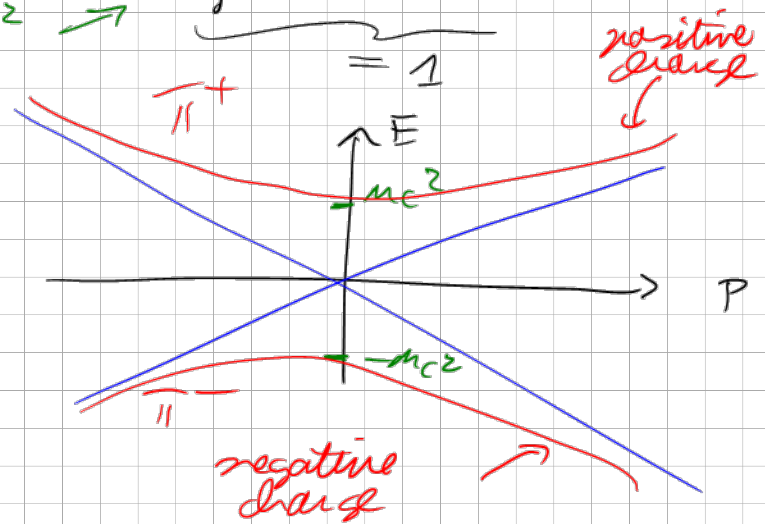
$$\frac{\partial}{\partial t}\rho + \mathrm{div}\, \vec{j} = 0 \quad \Rightarrow Q = \int d^3x\, \rho(\vec{x},t) = \text{constant}$$

$$Q = \frac{i\hbar}{2mc^2}\, 2\left(\mp \frac{i}{\hbar} mc^2\right) \int d^3x\, |\psi(\vec{x})|^2 = \overset{+mc^2}{\underset{-mc^2}{\pm}} \underbrace{\int d^3x\, |\psi(\vec{x})|^2}_{=1} = \pm 1$$

complex Klein-Gordon field:

charged pions $\pi^\pm$

$$\rho_{phys.} = \frac{i\hbar}{2mc^2}\, e \left\{\Psi^* \frac{\partial \Psi}{\partial t} - \Psi \frac{\partial \Psi^*}{\partial t}\right\}$$




Schrödinger:

$$Q = \int d^3x\, \psi^*(\vec{x})\, \psi(\vec{x})$$

scalar product: $\langle \psi, \phi \rangle = \int d^3x\, \psi^*(\vec{x})\, \phi(\vec{x})$

$\Rightarrow \quad Q = \langle \psi, \psi \rangle$

Klein-Gordon field:

$$Q = \int d^3x \, \frac{i\hbar e}{2mc^2} \left\{ \Psi^* \frac{\partial \Psi}{\partial t} - \Psi \frac{\partial \Psi^*}{\partial t} \right\} = \langle \Psi, \Psi \rangle$$

$$\langle \Psi, \Phi \rangle = \frac{i\hbar e}{2mc^2} \int d^3x \left\{ \Psi^* \frac{\partial \Psi}{\partial t} - \Psi \frac{\partial \Psi^*}{\partial t} \right\}$$

Hamilton equations:

$$\dot{q} = \frac{\partial H}{\partial p}$$

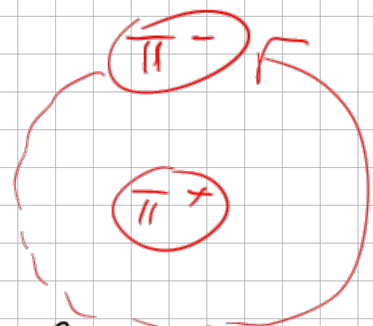
$$\dot{p} = -\frac{\partial H}{\partial q}$$

$$\underbrace{\vec{x} = \begin{pmatrix} q \\ p \end{pmatrix}}_{\text{phase space vector}} , \qquad \frac{d\vec{x}}{dt} = \underbrace{\left(\begin{array}{c|c} 0 & +1 \\ \hline -1 & 0 \end{array}\right)}_{\text{symplectic metric}} \underbrace{\begin{pmatrix} \frac{\partial}{\partial q} H \\ \frac{\partial H}{\partial p} \end{pmatrix}}_{= \vec{\nabla} H}$$

Minkowski metric:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \eta$$

## 12.4 Pionic Atom: Klein-Gordon Theory




- add minimal coupling
  covariant derivatives like in Schrödinger theory

$$i\hbar \frac{\partial}{\partial t} \longrightarrow i\hbar \frac{\partial}{\partial t} - q\varphi , \qquad \frac{\hbar}{i}\vec{\nabla} \longrightarrow \frac{\hbar}{i}\vec{\nabla} - q\vec{A}$$

$$\underbrace{\left(i\hbar\frac{\partial}{\partial t}-q\varphi\right)}_{\text{covariant derivative}}\left[\Psi(\vec{x},t)\,e^{\frac{iq}{\hbar}\Lambda(\vec{x},t)}\right] = e^{\frac{iq}{\hbar}\Lambda(\vec{x},t)}\left(i\hbar\frac{\partial}{\partial t}-q\underset{\sim}{\varphi'}\right)\Psi = 0$$

like in electrodynamics

$$\Rightarrow \left(i\hbar\frac{\partial}{\partial t}-q\varphi\right)^2\Psi = c^2\left(\frac{\hbar}{i}\nabla - q\vec{A}\right)^2\Psi + \frac{m^2c^4}{\hbar^2}\Psi$$

describes $\pi^{\pm}$ in presence of electromagnetic fields

specialize: $\vec{A}=\vec{0}$, $q=-e$, $\varphi(\vec{r}) = \frac{e}{4\pi\varepsilon_0 r}$

$$\left(i\hbar\frac{\partial}{\partial t} + \underbrace{\frac{e^2}{4\pi\varepsilon_0}}_{=\alpha\hbar c}\frac{1}{r}\right)^2\Psi = -\hbar^2c^2\Delta\Psi + (mc^2)^2\Psi$$

Sommerfeld fine structure constant: $\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c}$

separate space and time: $\Psi(\vec{x},t) = \psi(\vec{x})\,e^{-\frac{i}{\hbar}Et}$ (stationary state)

$$\left(E + \frac{\hbar c\alpha}{r}\right)^2\Psi(r,\vartheta,\varphi) = -\hbar^2c^2\left\{\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r} - \frac{\hat{L}^2}{\hbar^2r^2}\right\}\Psi(r,\vartheta,\varphi) + m^2c^4\Psi(r,\vartheta,\varphi)$$

another separation: $\Psi(r,\vartheta,\varphi) = f(r)\,Y_{\ell m}(\vartheta,\varphi)$

$$\left(E^2 + \frac{2\hbar c\alpha}{r}E + \frac{\hbar^2c^2\alpha^2}{r^2}\right)\overset{u(r)}{\cancel{f(r)}} = -\hbar^2c^2\left\{\frac{\partial^2}{\partial r^2} + \cancel{\frac{2}{r}\frac{\partial}{\partial r}} - \frac{\ell(\ell+1)}{r^2}\right\}\underset{u(r)}{\cancel{f(r)}} + m^2c^4\overset{u(r)}{\cancel{f(r)}} \;\Big|\cdot\frac{1}{2mc^2}$$

substitution: $f(r) = \frac{u(r)}{r}$

intermediate result:

$$\left\{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2} + \hbar^2\underline{\frac{\ell(\ell+1)-\alpha^2}{2mr^2}} - \underline{\frac{\alpha\hbar cE}{mc^2}\frac{1}{r}}\right\}u(r) = \frac{E^2-m^2c^4}{2mc^2}u(r)$$

radial Klein-Gordon equation $\overset{?}{=}$ Schrödinger equation

$$\left\{ -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2} + \underbrace{\hbar^2 \frac{l'(l'+1)}{2mr^2}}_{\text{centrifugal barrier}} - \underbrace{\frac{\hbar c\,\alpha'}{r}}_{\text{coulomb potential}} \right\} u(r) = E'\,u(r)$$

| | Schrödinger | | Klein-Gordon |
|---|---|---|---|
| $r^0$ | $E'$ | $\longleftrightarrow$ | $\frac{E^2-m^2c^4}{2mc^2}$ |
| $r^{-1}$ | $\alpha'$ | | $\alpha\frac{E}{mc^2}$ |
| $r^{-2}$ | $l'(l'+1)$ | $\longleftrightarrow$ | $l(l+1)-\alpha^2$ |

Schrödinger solution:

$$E' = -\frac{1}{2}mc^2\,\alpha'^2\,\frac{1}{n'^2}\,, \qquad n' = \underbrace{n_r'}_{=n_r} + (l'+1)$$

1) $l'(l'+1) \overset{!}{=} l(l+1)-\alpha^2$

$\Rightarrow\ l' = -\frac{1}{2} + \sqrt{(l+\frac{1}{2})^2-\alpha^2}$

$l$ integer $\Rightarrow$ $l'$ is no longer integer

$n_r = n_r' = 0, 1, 2, \dots, n-l-1$ integer but $n'$ not integer

2) $$\frac{E^2 - m^2c^4}{2mc^2} = -\frac{1}{2} mc^2 \alpha^2 \frac{E^2}{(mc^2)^2} \cdot \frac{1}{\left[n_r + \frac{1}{2} + \sqrt{(l+\frac{1}{2})^2 - \alpha^2}\,\right]^2}$$

$$\Rightarrow E = \frac{mc^2}{\sqrt{1 + \dfrac{\alpha^2}{\left[n_r + \frac{1}{2} + \sqrt{(l+\frac{1}{2})^2 - \alpha^2}\,\right]^2}}}$$

$$n' \approx n_r + \frac{1}{2}\left(l + \frac{1}{2}\right) - \frac{\alpha^2}{2(l+\frac{1}{2})} + \ldots$$

small $\alpha \approx \frac{1}{137} \ll 1$

$$E \approx Mc^2 \left\{ \underbrace{1}_{\text{rest energy}} - \underbrace{\frac{\alpha^2}{2n^2}}_{\text{schrödinger hydrogen}} - \frac{\alpha^4}{2n^4}\left[\frac{n}{l + \frac{1}{2}} - \frac{3}{4}\right] + \ldots \right\}$$

schrödinger frustrated

$$E_{Dirac} \approx mc^2 \left\{ 1 - \frac{\alpha^2}{2n^2} \underbrace{- \frac{\alpha^4}{2n^4}\left[\frac{n}{j + \frac{1}{2}} - \frac{3}{4}\right]}_{\text{special relativistic corrections}} + \ldots \right\}$$

Klein-Gordon prediction fails to explain measured fine structure corrections due to missing spin 1/2