Status:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1$$

$$\Box h_{\mu\nu} + \partial_\mu \partial_\nu h_\lambda{}^\lambda - \partial_\lambda \partial_\nu h^\lambda{}_\mu - \partial_\mu \partial_\lambda h^\lambda{}_\nu = -\frac{16\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right)$$

$$x^\mu \to x'^\mu = x^\mu + \varepsilon^\mu(x)$$

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \varepsilon_\nu - \partial_\nu \varepsilon_\mu$$

$$\Box \varepsilon_\nu = \partial_\mu h_\nu{}^\mu - \frac{1}{2} \partial_\nu h_\mu{}^\mu \qquad \begin{cases} \text{Hilbert gauge: } \partial_\nu h'^{\ \mu}_\mu = 2\partial_\mu h'^{\ \mu}_\nu \\ \Box h'_{\mu\nu} = -\frac{16\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right) \end{cases}$$

3.1.5 Plane Wave:

$$\Box h_{\mu\nu}(x) = 0 \Rightarrow h_{\mu\nu}(x) = \underbrace{e_{\mu\nu}}_{\text{polarization tensor}} e^{-ik^\lambda x_\lambda} + \cancel{c.c.} \Rightarrow k^\lambda k_\lambda = 0 \Rightarrow \omega = c|\vec{k}|$$

neglect for simplicity

$$\partial_\nu h_\mu{}^\mu = 2\partial_\mu h'_\nu{}^\mu \Rightarrow k_\nu e_\mu{}^\mu = 2 e_{\nu\mu} k^\mu \quad (*)$$

intermediate result: polarization tensor $e_{\mu\nu} = e_{\nu\mu}$

10 degrees of freedom − 4 Hilbert gauge conditions

$\longrightarrow$ 6 degrees of freedom

$$(e_{\mu\nu}) = \begin{pmatrix} \cancel{e_{00}} & -\cancel{e_{13}} & -\cancel{e_{23}} & -\cancel{\frac{e_{00}+e_{33}}{2}} \\ -\cancel{e_{13}} & e_{11} & e_{12} & \cancel{e_{13}} \\ -\cancel{e_{23}} & e_{12} & -e_{11} & \cancel{e_{23}} \\ -\cancel{\frac{e_{00}+e_{33}}{2}} & \cancel{e_{13}} & \cancel{e_{23}} & \cancel{e_{33}} \end{pmatrix}$$

To show: Choose homogeneous solution for $\Box \varepsilon_\mu = 0$ allows to get rid of those coefficients.

Choose: $\varepsilon_\mu(x) = \underbrace{\delta_\mu}_{\text{four vector amplitude}} e^{-ik^\lambda x_\lambda}$

$\Rightarrow$ regauging of polarization tensor

$h'_{\mu\nu} = e'_{\mu\nu}\, e^{-ik^\lambda x_\lambda}$

$e'_{\mu\nu} = e_{\mu\nu} + i(k_\mu \delta_\nu + k_\nu \delta_\mu)$; $(k^\lambda) = k \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, $(k_\lambda) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} k$

propagation in x^3-direction

$$(e'_{\mu\nu}) = (e_{\mu\nu}) + ik \begin{pmatrix} 2\delta_0 & \delta_1 & \delta_2 & \delta_3 - \delta_0 \\ \delta_1 & 0 & 0 & -\delta_1 \\ \delta_2 & 0 & 0 & -\delta_2 \\ \delta_3 - \delta_0 & -\delta_1 & -\delta_2 & -2\delta_3 \end{pmatrix}$$

$e'_{00} = e_{00} + 2ik\delta_0 \overset{!}{=} 0 \Rightarrow \delta_0 = -i\frac{e_{00}}{2k}$

$e'_{13} = e_{13} - ik\delta_1 \overset{!}{=} 0 \Rightarrow \delta_1 = i\frac{e_{13}}{k}$

$$e'_{23} = e_{23} - ik\delta_2 \overset{!}{=} 0 \Rightarrow \delta_2 = i\frac{e_{23}}{k}$$

$$e'_{33} = e_{33} - 2ik\delta_3 \overset{!}{=} 0 \Rightarrow \delta_3 = i\frac{e_{33}}{2k}$$

Plane gravitational wave propagating in x^3-direction

$$\left(h_{\mu\nu}(\vec{x},t)\right) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e_{11} & e_{12} & 0 \\ 0 & e_{12} & -e_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{ik(x^3 - ct)} + c.c. \quad (*)$$

$$\begin{pmatrix} k \\ 0 \\ 0 \\ k \end{pmatrix}$$

$e_{\mu\nu} k^\nu = 0 \quad \hat{=}$ transversality condition for gravitational wave

|| Hilbert gauge

$$\frac{1}{2} k_\mu \underbrace{e_\lambda{}^\lambda}_{= e_{00} - (e_{11} + e_{22} + e_{33})} = 0$$

$e_{11} = 0$ or $e_{12} = 0$ correspond to two linearly polarised gravitational waves.

3.1.6 Helicity:

Investigate rotation around x^3-direction

$$(\Lambda^{\mu}{}_{\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi & 0 \\ 0 & -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$e' = \Lambda^T e \Lambda \quad \Rightarrow \quad e'_{\mu\nu} = \Lambda^{\mu'}{}_{\mu}\, e_{\mu'\nu'}\, \Lambda^{\nu'}{}_{\nu}$$

$$e' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e'_{11} & e'_{12} & 0 \\ 0 & e'_{12} & -e'_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} e'_{11} \\ e'_{12} \end{pmatrix} = \begin{pmatrix} \cos(2\varphi) & -\sin(2\varphi) \\ \sin(2\varphi) & \cos(2\varphi) \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{22} \end{pmatrix}$$

$$e_{\pm} = e_{11} \pm i e_{12} \stackrel{a}{=}$$

$$e_{11} = \frac{1}{2}(e_+ + e_-)$$

$$e_{12} = \frac{-i}{2}(e_+ - e_-)$$

$$\Rightarrow e'_{\pm} = e^{\pm 2 i \varphi}\, e_{\pm}$$ $\Rightarrow$ helicity 2 (graviton)

Maxwell: $e^{\pm 1 i \varphi}$ $\Rightarrow$ helicity 1 (photon)

Weyl: $e^{\pm \frac{1}{2} i \varphi}$ $\Rightarrow$ helicity $\frac{1}{2}$ (neutrinos according to previous theories)

3.2 Particles in Field of Gravitational Waves:

How does a wave (*) affects particles? Look at geodesics:

$$\frac{d^2x^\lambda}{d\tau^2} = -\Gamma_{\mu\nu}{}^{\lambda}\,\frac{dx^\mu}{d\tau}\,\frac{dx^\nu}{d\tau}$$

$$\Gamma_{\mu\nu}{}^{\lambda} = g^{\lambda\kappa}\,\Gamma_{\mu\nu\kappa}\,, \qquad \Gamma_{\mu\nu\kappa} = \frac{1}{2}\left(\partial_\mu h_{\nu\kappa} + \partial_\nu h_{\kappa\mu} - \partial_\kappa h_{\mu\nu}\right)$$

initial condition: $\left.\frac{dx^i}{d\tau}\right|_{\tau=0} = 0 \;\hat{=}\;$ particles are initially at rest

$$\left.\frac{d^2x^i}{d\tau^2}\right|_{\tau=0} = -\Gamma_{\mu\nu}{}^{i}\Big|_{\tau=0}\left.\frac{dx^\mu}{d\tau}\right|_{\tau=0}\left.\frac{dx^\nu}{d\tau}\right|_{\tau=0} = -\Gamma_{00}{}^{i}\Big|_{\tau=0}\left(\left.\frac{dx^0}{d\tau}\right|_{\tau=0}\right)^2 \equiv 0$$

particles remain at rest despite gravitational wave

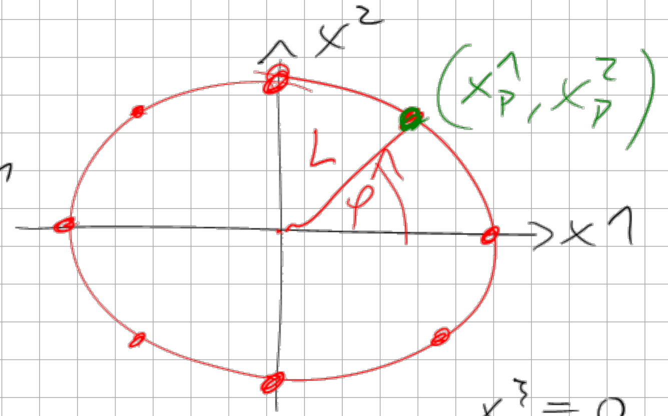
$$\Gamma_{00}{}^{i}\Big|_{\tau=0} = g^{i\lambda}\,\Gamma_{00\lambda}\Big|_{\tau=0} = g^{ij}\,\Gamma_{00j}\Big|_{\tau=0}$$

$$\Gamma_{00j} = \frac{1}{2}\Big(\partial_0 \underbrace{h_{0j}}_{=0} + \partial_0 \underbrace{h_{j0}}_{=0} - \underbrace{\partial_j h_{00}}_{=0}\Big) \equiv 0$$

Result: The particles stay where they are, but the distances between them change.

$$(x^1)^2 + (x^2)^2 = L^2$$

real position



proper length:

$$ds^2 = g_{\mu\nu}\,dx^\mu dx^\nu = c^2dt^2 - dl^2 - (dx^3)^2$$

$$x^3 = 0$$

$$dl^2 = \left[\delta_{mn} - h_{mn}(t)\right] dx^m dx^n; \quad m, n = 1, 2$$

$$\left(h_{mn}(x^3=0,t)\right) = \begin{pmatrix} e_{11} & e_{12} \\ e_{12} & -e_{11} \end{pmatrix} e^{-i\omega t} \boxed{+\ c.c.}$$ independent of x^1, x^2

Point (x_p^1, x_p^2): distance of point from origin

$$\rho^2 = [\delta_{mn} - h_{mn}(t)]\, x_p^m x_p^n \ ; \quad m,n = 1,2$$

$$x_p^1 = L\cos\varphi, \quad x_p^2 = L\sin\varphi$$

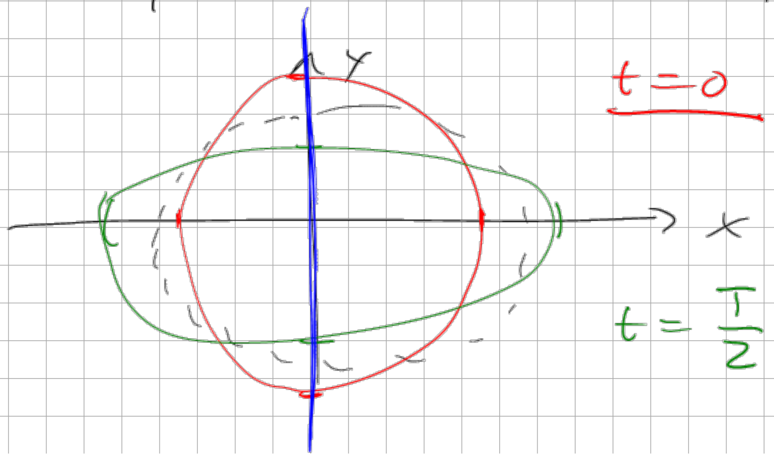
$$\rho^2 = (x_p^1)^2 + (x_p^2)^2 - (x_p^1)^2 e_{11}\, 2\cos\omega t + (x_p^2)^2 e_{11}\, 2\cos\omega t$$
$$- 2 x_p^1 x_p^2\, e_{12}\, 2\cos\omega t$$
$$= L^2 \left\{ 1 - 2\cos\omega t \left[\underbrace{(\cos^2\varphi - \sin^2\varphi)}_{=\cos(2\varphi)} e_{11} + \underbrace{2\sin\varphi\cos\varphi}_{=\sin(2\varphi)} e_{12} \right] \right.$$

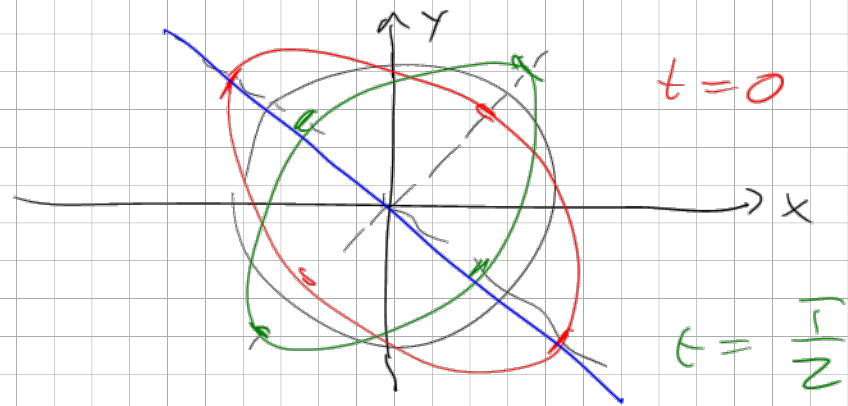
$e_{11} = h, \ e_{12} = 0$

$$\rho^2 = L^2 \{ 1 - 2h\cos(2\varphi) \cos \overset{=\frac{2\pi}{T}}{\omega} t \}$$

$e_{11} = 0, \ e_{12} = h$

$$\rho^2 = L^2 \{ 1 - 2h \sin(2\varphi) \cos\omega t \}$$





- $x = \rho \cos \varphi$, $y = \rho \sin \varphi$

  Illustration for distances from origin

- As we will see : $(\rho - L)/L \mathrel{\hat{=}} 10^{-20}$

  very tiny deformation $\Rightarrow$ figure is overexaggeration.

- Both polarization directions have an angle of $\pi/4$ due to helicity 2.

In summary: distances between matter point change

$\Rightarrow$ quadrupol moment of matter

Conversely this means that a quadrupole moment of matter

would lead to a gravitational wave

3.3 Quadrupole Radiation of Gravitational Wave