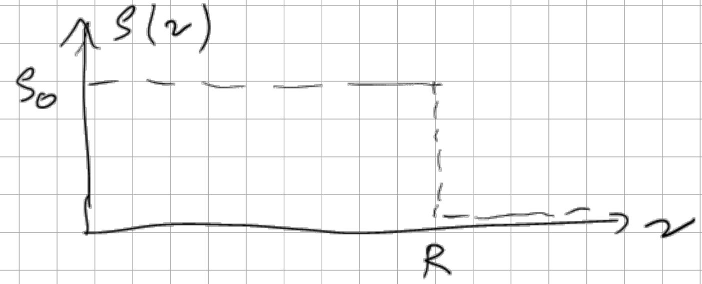
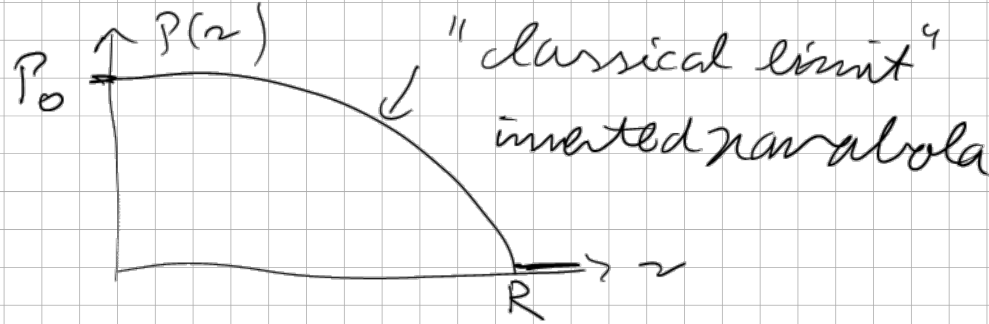
Reminder:

nonrelativistic star equilibrium: gravity force = pressure force

$$\frac{dP(r)}{dr} = -\frac{G}{r^2}\underbrace{M(r)}_{=4\pi\int_0^r dr'\,\rho(r')\,r'^2}\rho(r)$$

incompressible matter:





4.2 Inner Schwarzschild Metric:

Aim: describe spacetime metric within star which has an isotropic mass distribution and is in equilibrium

$$c^2 d\tau^2 = ds^2 = \underbrace{B(r)}_{\text{gravitational red shift}} c^2 dt^2 - \underbrace{A(r)}_{\text{"sees" star volume}} dr^2 - r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)$$

vacuum: $R_{\mu\nu} = 0$

matter: $R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{T}{2}\, g_{\mu\nu}\right)$

ideal fluid: $T_{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) u_\mu u_\nu - P g_{\mu\nu}$

local conservation law: $D_\nu T^{\mu\nu} = 0$

static: $u^i = 0 = u_i$

general condition: $c^2 = g_{\mu\nu} u^\mu u^\nu = g_{00} (u^0)^2 \Rightarrow u^0 = \frac{c}{\sqrt{B(r)}}, \quad u_0 = c\sqrt{B(r)}$

$$(T_{\mu\nu}) = \mathrm{diag}\left(\rho c^2 B, P A, P r^2, P r^2 \sin^2\vartheta\right)$$

$$(g_{\mu\nu}) = \mathrm{diag}\left(B, -A, -r^2, -r^2 \sin^2\vartheta\right)$$

$$(g^{\mu\nu}) = \mathrm{diag}\left(\frac{1}{B}, -\frac{1}{A}, -\frac{1}{r^2}, -\frac{1}{r^2 \sin^2\vartheta}\right)$$

$$T = T_\mu{}^\mu = T_{\mu\nu} g^{\mu\nu} = \rho \cdot c^2 - 3P$$

$$R_{00} = \frac{B''}{2A} - \frac{B'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{B'}{rA} = \frac{4\pi G}{c^4}\left(\rho c^2 + 3P\right) B$$

$$R_{11} = -\frac{B''}{2B} + \frac{B'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{A'}{rA} = \frac{4\pi G}{c^4}\left(\rho c^2 - P\right) A$$

$$R_{22} = 1 + \frac{r}{2A}\left(\frac{A'}{A} - \frac{B'}{B}\right) - \frac{1}{A} = \frac{4\pi G}{c^4}\left(\rho c^2 - P\right) r^2$$

$R_{33} \hat{=} R_{22}$

$$\frac{R_{00}}{2B} + \frac{R_{11}}{2A} + \frac{R_{22}}{r^2} = \frac{A'}{rA^2} - \frac{1}{r^2 A} + \frac{1}{r^2} = \frac{8\pi G}{c^4}\rho c^2 \qquad (1)$$

$\Rightarrow$ This determines $A(r)$:

$$A(r) = \frac{1}{1-\frac{2G}{c^2 r}\mathcal{M}(r)} \qquad (2) \qquad \overset{r>R}{\Longrightarrow} \qquad A(r) = \frac{1}{1-\frac{2G}{c^2 r}M}$$

Evaluate energy-momentum conservation for $\mu=1$:

$$0 = D_\nu T^{1\nu} = \partial_\nu T^{1\nu} + \Gamma^1_{\nu\lambda} T^{\lambda\nu} + \Gamma^\nu_{\nu\lambda} T^{1\lambda}$$

$$0 = \partial_\nu\left[\cancel{\left(\rho+\frac{P}{c^2}\right)u^1u^\nu} - P g^{1\nu}\right] + \Gamma^1_{\nu\lambda}\left[\left(\rho+\frac{P}{c^2}\right)u^\lambda u^\nu - P g^{\lambda\nu}\right] + \Gamma^\nu_{\nu\lambda}\left[\cancel{\left(\rho+\frac{P}{c^2}\right)u^1u^\lambda} - P g^{1\lambda}\right]$$

$$0 = -\underbrace{(\partial_\nu P) g^{1\nu}}_{\frac{dP}{dr}g^{11}} - P\underbrace{\left(\partial_\nu g^{1\nu} + \Gamma^1_{\nu\lambda}g^{\lambda\nu} + \Gamma^\nu_{\nu\lambda}g^{1\lambda}\right)}_{= D_\nu g^{1\nu} \equiv 0} + \underbrace{\boxed{\Gamma^1_{00}}}_{\frac{B'}{2A}}\left(\rho+\frac{P}{c^2}\right)(u^0)^2$$

due to metricity condition

$$\Rightarrow \quad \frac{B'}{B} = -\frac{2P'}{\rho c^2+P} \qquad (3)$$

(1) — (3) inserted into R_{22} equation:

$$\frac{dP}{dr} = -\underbrace{\frac{GM\rho}{r^2}}_{\text{classical part}} \frac{\left(1+\frac{P}{\rho c^2}\right)\left(1+\frac{4\pi P r^3}{M c^2}\right)}{1-\frac{2GM}{c^2 r}}\,; \qquad M(r) = 4\pi\int_0^r dr'\, r'^2 \rho(r')$$

→ differential equation determining the star pressure $P(r)$

Tolman – Oppenheimer – Volkoff equation

Integration of (2) then yields

$$B(r) = \exp\left\{-\frac{2G}{c^2}\int_r^\infty \frac{dr'}{r'^2}\, \frac{\overset{M}{\cancel{M(r')}} + \cancel{\frac{4\pi P(r') r'^3}{c^2}}}{1-\frac{2G\,\cancel{M(r')}\,M}{c^2 r'}}\right\}$$

valid for all r, inside and outside the star

Extrapolate this outside of star:

$$\underbrace{\rho(r) = 0}_{} = P(r) \qquad r \geq R$$

$$M(r) = \underbrace{M(R) = M}_{}$$

Evaluate $B(r)$ for $r \geq R$:

$$B(r) = \exp\left\{-\frac{2GM}{c^2}\int_r^\infty \frac{dr'}{r'^2}\, \frac{1}{\underbrace{\boxed{1-\frac{2GM}{c^2 r'}}}_{=\,x(r)}}\right\}$$

$$= \exp\left\{ -\underbrace{\int_{x(r)}^{1} \frac{dx'}{x'}}_{\ln 1 - \ln x(r)} \right\} = \boxed{x(r) = 1 - \frac{2GM}{c^2 r}} \qquad r > R$$

Vacuum solution gives this $B(r)$ with M as integration constant

Physical identification of M by nonrelativistic limit of geodetic equation and comparing with Newton

Here: M comes automatically out to be the gravitational mass outside of the star.

Remark: Oppenheimer - Volkoff equation

$$\begin{cases} P'(r) = P'\left(P(r), \rho(r), M(r) \right) \\ M'(r) = 4\pi \rho(r)\, r^2 \end{cases}$$

2 equation for 3 parameters

unique solution necessitates one additional equation

$\Rightarrow$ equation of state $\quad P(r) = P\left(\rho(r) \right)$

polytrope equation of state $\quad P = K \cdot \rho^{\gamma}$ γ polytrope index

incompressible fluid: $\gamma \to \infty$ $\quad P = K' \left(\frac{\rho}{\rho_0} \right)^{\gamma}$

Remark: so far no temperature

$\rightarrow$ $T(r)$ wanted

$\rightarrow$ one additional equation is needed

caloric equation of state $\quad E = (\rho(r), T(r))$

4.3 Relativistic star: incompressible fluid:

$$\rho(r) = \begin{cases} \rho & ; r \le R \\ 0 & ; r > R \end{cases}, \quad M(r) = 4\pi \int_0^r dr' \rho(r') r'^2 = \begin{cases} \frac{4\pi}{3}\rho_0 r^3 = M\left(\frac{r}{R}\right)^3 & r \le R \\ \frac{4\pi}{3}\rho_0 R^3 = M & r \ge R \end{cases}$$

Oppenheimer – Volkoff equation: $\quad r \le R$

$$P'(r) = -\frac{G M(r)\rho_0}{r^2} \cdot \frac{\frac{P + \rho_0 c^2}{\rho c^2} \cdot \frac{P + \frac{Mc^2}{4\pi r^3}}{M c^2/4\pi r^3}}{1 - \frac{2GM}{c^2 r}}$$

$$\frac{dP}{(P + \rho_0 c^2)(P + \frac{\rho_0 c^2}{3})} = -\frac{3}{2c^2\rho_0} \frac{x\,dx}{1-x^2}$$

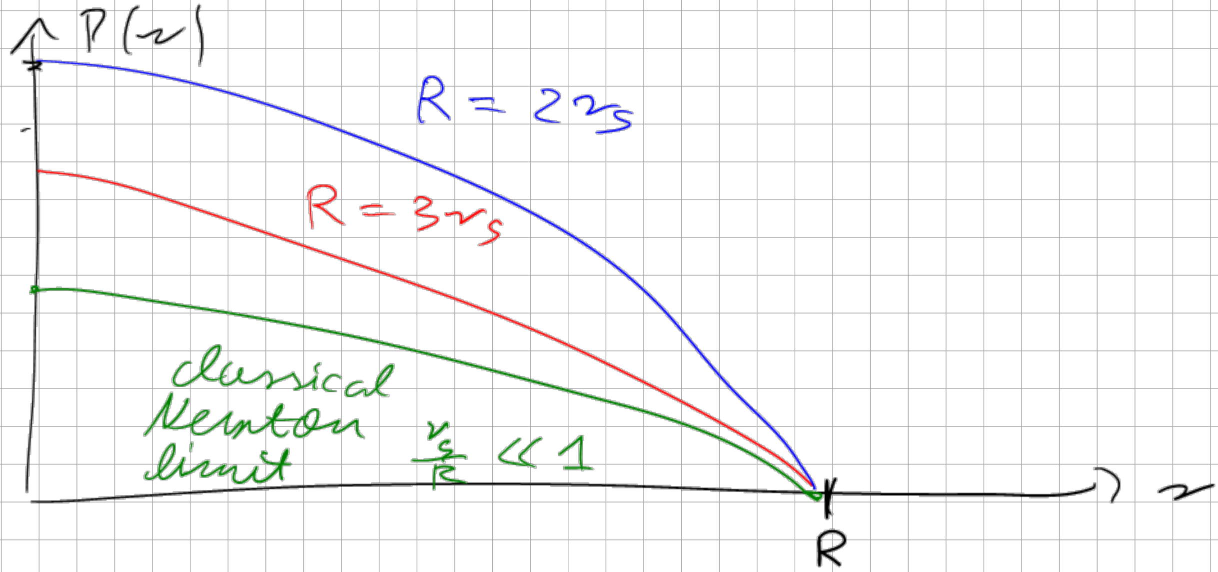
$$x = \sqrt{\frac{8\pi G \rho_0}{3c^2}}\, r$$

$\Rightarrow$ can be integrated!

$$P(r) = \rho_0 c^2 \frac{\sqrt{1-\frac{r_s r^2}{R^3}} - \sqrt{1-\frac{r_s}{R}}}{3\sqrt{1-\frac{r_s}{R}} - \sqrt{1-\frac{r_s r^2}{R^3}}} \qquad 0 \le r \le R$$

1) $P(R) = 0$

2) classical limit $\frac{r_s}{R} \ll 1$: $P(r) = \boxed{\rho_0 c^2 \frac{r_s}{4R}\left(1-\frac{r^2}{R^2}\right)}$



Check for divergency of $P(r)$:

$$P(0) = \frac{1-\sqrt{1-\frac{r_s}{R}}}{3\sqrt{1-\frac{r_s}{R}} - 1} = \infty \Rightarrow 3\sqrt{1-\frac{r_s}{R}} = 1 \Rightarrow R > \frac{9}{8} r_s = R_{crit}$$

R given: $M < \frac{8}{9}\frac{c^2}{G} R$ mass upper limit

M given: $R > \frac{9}{8}\frac{GM}{c^2}$ radius lower bound

- relativistic effect, independent of physical nature of incompressible matter

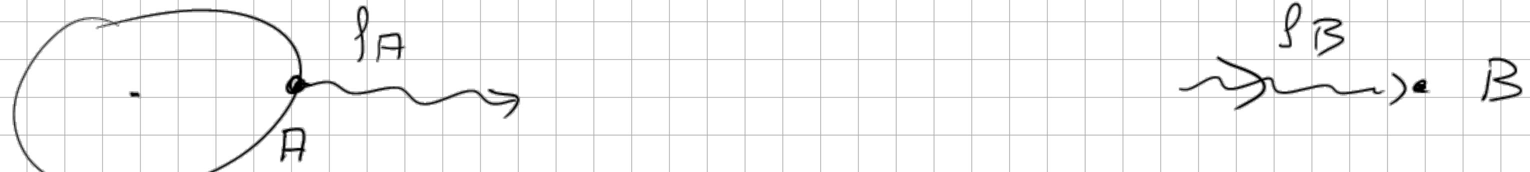
$\longrightarrow$ gravitational collapse

$\longrightarrow$ time dependence needed in metric

Coefficients in Schwarzschild metric:

$$1)\quad B(r)=\begin{cases}\frac{1}{4}\left[3\sqrt{1-\frac{2S}{R}}-\sqrt{1-\frac{2Sr^2}{R^3}}\right]^2 & ;\ r\leq R\\ 1-\frac{2S}{r} & ;\ r\geq R\end{cases}\quad \text{continuous}$$

$\longrightarrow$ gravitational redshift



$g_{00}(r=R)=B(R)$ $\qquad\qquad$ $g_{00}(r=\infty)=1$

$\underbrace{d\tau_A}_{\text{period in } A}=\sqrt{g_{00}(r=R)}\,dt_A$ $\qquad\qquad$ $\underbrace{d\tau_B}_{=\text{period in } B}=\sqrt{g_{00}(r=\infty)}\,dt_B$

$= 0$

red shift parameter $\quad z=\frac{f_A}{f_B}-1=\frac{\lambda_B}{\lambda_A}-1=\sqrt{\frac{g_{00}(r=\infty)}{g_{00}(r=R)}}-1$

$=\frac{1}{\sqrt{1-\frac{2S}{R}}}-1\ \underset{\uparrow}{<}\ 2$ limit for gravitational

$R' > \frac{9}{8} r_s$ red shift

larger red shifts are possible:

- light observed from inside star
- radiation stems from region $R \simeq r_s \implies$ gravitational collapse

largest $z \approx 5$ (quasars) no gravitational red shift but a cosmological red shift due to expansion of universe

2) $V = \int_{r \leq R} dV = \int_0^R dr \int_0^{\pi} d\vartheta \int_0^{2\pi} d\varphi \underbrace{\sqrt{|g_{11} g_{22} g_{33}|}}_{r^2 \sin\vartheta \frac{1}{\sqrt{1 - \frac{r_s}{R^3} r^2}}}$

$= \frac{4\pi}{3} R^3 \; F\left(X = \sqrt{\frac{r_s}{R}}\right)$

$F(X) = \frac{3}{2X^3}\left(\arcsin X - X\sqrt{1 - X^2}\right) > 1$

$= \begin{cases} 1 + \frac{3}{10}\frac{r_s}{R} + \ldots, & X << \quad \text{Newton limit} \\ 1.64, & X = \sqrt{\frac{8}{9}} \end{cases}$

mass defect: $\Delta M = V \rho_0 - M > 0$

$\Delta M c^2 = - \frac{E_{grav}}{c^2}$