3.3 Quadrupole Radiation of Gravitational Waves

Basic idea:

last time: a gravitational wave leads to a change of quadrupole moment of mass

therefore: conversely, a change of a quadrupole moment of mass should create gravitational waves

3.3.1 Inhomogeneous Wave Equation:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \; ; \quad |h_{\mu\nu}| \ll 1$$

Hilbert gauge: $2\,\partial_\mu h^\mu{}_\nu = \partial_\nu h^\mu{}_\mu$

$$\Box h_{\mu\nu} = -\frac{16\pi G}{c^4}\left(T_{\mu\nu} - \frac{T}{2}\eta_{\mu\nu}\right) ; \quad T = T^\mu{}_\mu$$

particular solution with Green function:

$$\Box \underbrace{G(\vec{x}, t; \vec{x}', t')}_{= G(\vec{x}-\vec{x}',\, t-t')} = \delta(\vec{x}-\vec{x}')\,\delta(t-t')$$

$$= G(\vec{x}-\vec{x}',\, t-t') \qquad \text{homogeneity: space, time}$$

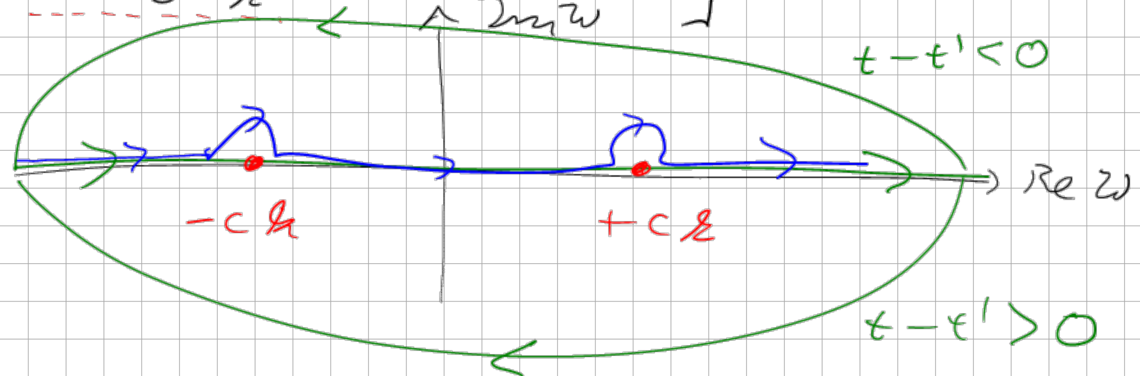
$$= \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, G(\vec{k},\omega)\, e^{i[\vec{k}(\vec{x}-\vec{x}') - \omega(t-t')]}$$

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta$$

$$\left(-\frac{\omega^2}{c^2} + \vec{k}^2\right) G(\vec{k},\omega) = 1 \Rightarrow G(\vec{k},\omega) = \frac{-c^2}{\omega^2 - c^2\vec{k}^2}$$

$\omega = c|\vec{k}|$

$$G(\vec x-\vec x', t-t') = -c^2\int\frac{d^3k}{(2\pi)^3}\left[\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,\frac{1}{\omega^2-c^2k^2}\,e^{-i\omega(t-t')}\right]e^{i\vec k\cdot(\vec x-\vec x')}$$



here: causal propagator

$$\left|e^{-i\omega(t-t')}\right| = e^{\mathrm{Im}\,\omega\,(t-t')}$$

$$\not\to\infty\quad\begin{cases} t-t'>0: & \mathrm{Im}\,\omega<0\\ t-t'<0: & \mathrm{Im}\,\omega>0\end{cases}$$

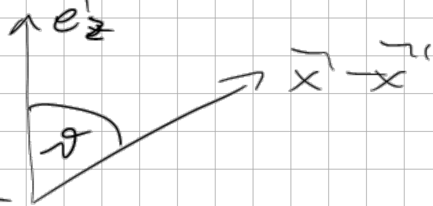
$$G(\vec x-\vec x', t-t') = 0 \quad \text{for} \quad t-t'<0 \;\hat{=}\; \text{causality}$$

$$\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,\frac{e^{-i\omega(t-t')}}{\omega^2-c^2k^2} = \frac{1}{2\pi}(-2\pi i)\left\{\underbrace{\operatorname*{Res}_{\omega=ck}}_{=\lim_{\omega\to ck}\cancel{(\omega-ck)}}\frac{e^{-i\omega(t-t')}}{\cancel{(\omega-ck)}(\omega+ck)} + \underbrace{\operatorname*{Res}_{\omega=-ck}}_{=\lim_{\omega\to -ck}\cancel{(\omega+ck)}}\frac{e^{-i\omega(t-t')}}{(\omega-ck)\cancel{(\omega+ck)}}\right\}$$

$$= -\frac{i}{2\pi}\left\{\frac{1}{2ck}e^{-ick(t-t')} - \frac{1}{2ck}e^{+ick(t-t')}\right\} = \frac{-1}{ck}\sin[ck(t-t')]$$

$$G(\vec x-\vec x', t-t') = c\int\frac{d^3k}{(2\pi)^3}\,\frac{\sin[ck(t-t')]}{k}\,e^{i\vec k\cdot(\vec x-\vec x')}$$

$$= \frac{c}{(2\pi)^3}\,2\pi\,\frac{1}{2}\int_{\cancel{0}\;-\infty}^{\infty}dk\,\cancel{k^2}\,\underbrace{\int_0^{\pi}d\vartheta\,\sin\vartheta\; e^{ik\cos\vartheta\,|\vec x-\vec x'|}}_{u(\vartheta)=\cos\vartheta\;=\;\frac{2}{\cancel{k}}\,\frac{\sin k|\vec x-\vec x'|}{|\vec x-\vec x'|}}\;\frac{\sin ck(t-t')}{\cancel{k}}$$



$$= -\frac{\cancel{c}}{4\pi|\vec{x}-\vec{x}'|}\left\{\underbrace{\delta\left(|\vec{x}-\vec{x}'| - c(t-t')\right)}_{= \cancel{\frac{1}{c}}\,\delta\left(t-t'-\frac{1}{c}|\vec{x}-\vec{x}'|\right)} + \cancel{\delta\left(\underbrace{|\vec{x}-\vec{x}'| + c(t-t')}_{>0 \text{ for } t>t'}\right)}\right\}$$

green function of Poisson equation

finite propagation of signals with c

Result: particular solution of inhomogeneous wave equation

$$h_{\mu\nu}(\vec{x},t) = -\frac{4G}{c^4}\int d^3x' \frac{1}{|\vec{x}-\vec{x}'|} S_{\mu\nu}\left(\vec{x}', t - \underbrace{\frac{1}{c}|\vec{x}-\vec{x}'|}_{\text{due to retardation}}\right)$$

$S_{\mu\nu} = T_{\mu\nu} - \frac{T}{2}\eta_{\mu\nu}$: source term

3.3.2 Asymptotic Fields:

assumption: time periodic, spatially confined mass distribution

$$T_{\mu\nu}(\vec{x},t) = T_{\mu\nu}(\vec{x})\, e^{-i\omega t} + c.c. \quad \begin{cases} \neq 0 \,; & |\vec{x}| \leq R \\ = 0 \,; & |\vec{x}| > R \end{cases}$$

$$h_{\mu\nu}(\vec{x},t) = -\frac{4\pi}{c^4}\boxed{e^{-i\omega t}}\int d^3x' S_{\mu\nu}(\vec{x}')\,\frac{\boxed{e^{+ik|\vec{x}-\vec{x}'|}}}{\boxed{|\vec{x}-\vec{x}'|}}$$

up to 1st order

spherical wave

up to 0th order

$k = \frac{\omega}{c}$

far field: $\underbrace{|\vec{x}|}_{\text{observer}} \gg \underbrace{|\vec{x}'|}_{\text{origin of wave}}$

$$|\vec{x}-\vec{x}'| = \sqrt{r^2 - 2\vec{x}\cdot\vec{x}' + r'^2} = r\left(1 + \frac{\vec{x}\cdot\vec{x}'}{r^2} + \dots\right)$$

$$h_{\mu\nu}(\vec{x},t) = -\frac{4G}{c^4}\underbrace{\frac{e^{i(kr-\omega t)}}{r}}_{\text{spherical wave}} \int d^3x' \, S_{\mu\nu}(\vec{x}') \, e^{i \underbrace{\boxed{k\frac{\vec{x}}{r}}}_{=\vec{k}} \cdot \vec{x}'}$$

$$|\vec{k}| = k$$

four wave vector: $(k^\lambda) = \begin{pmatrix} \omega/c \\ \vec{k} \end{pmatrix} = \begin{pmatrix} \omega/c \\ k\vec{x}/r \end{pmatrix}$ $\quad (x^\lambda) = \begin{pmatrix} ct \\ \vec{x} \end{pmatrix}$

$$k^\lambda x_\lambda = \omega t - kr$$

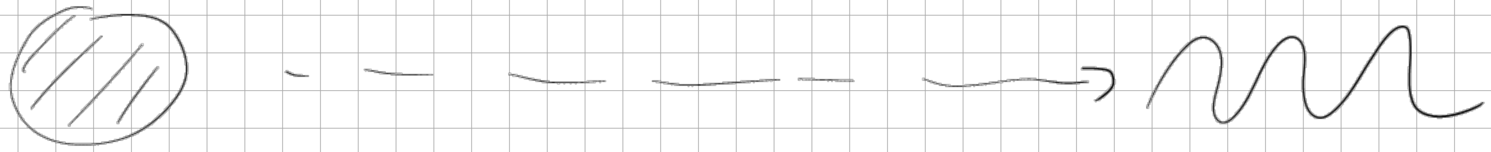
$$h_{\mu\nu}(x^\lambda) = \underbrace{e_{\mu\nu}}_{\text{polarization tensor}} \, e^{-i k^\lambda x_\lambda}$$

$$= -\frac{4G}{c^4 r} S_{\mu\nu}(\vec{k})$$

In far field the spherical wave is approximately a planar wave

Question: What is now the energy-momentum of a gravitational wave?

3.3.3 Energy-Momentum Tensor for a Planar Gravitational Waves



source — planar gravitational wave

periodic change
of mass quadrupole
moment

Einstein field equation:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \Rightarrow T_{\mu\nu} = \frac{c^4}{8\pi G}\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \underbrace{R}_{= g^{\lambda\kappa} R_{\lambda\kappa}}\right)$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1$$

$R^{(0)}_{\mu\nu} = 0$ due to flat Minkowski space-time

$R^{(1)}_{\mu\nu} = 0 \Rightarrow h_{\mu\nu}$ solves homogeneous wave equations

$$g^{\lambda\kappa} = \eta^{\lambda\kappa} - h^{\lambda\kappa}$$

$$g_{\mu\nu}\, g^{\nu\lambda} = \ldots = \delta_\mu{}^\lambda + O(h^2)$$

$$t^{grav}_{\mu\nu} = \frac{c^4}{16\pi G}\left\{ 2 R^{(2)}_{\mu\nu} - \left[(\eta_{\mu\nu} + h_{\mu\nu})(\eta^{\lambda\kappa} + h^{\lambda\kappa})\, R_{\lambda\kappa}\right]^{(2)} \right\}$$

$$\overbrace{\xi_{\mu\nu}\,\xi^{\lambda\kappa}\,\underbrace{R^{(2)}_{\lambda\kappa}}_{\sim k_\lambda k_\kappa}}$$

$$R_{\mu\nu} = R_{\lambda\mu\nu}{}^{\lambda} = g^{\lambda\kappa}\, R_{\lambda\mu\nu\kappa}$$

$$\underbrace{\qquad}_{\lambda^\lambda k_\lambda \equiv 0}$$

$$R^{(2)}_{\mu\nu} = \xi^{\lambda\kappa}\, R^{(2)}_{\lambda\mu\nu\kappa} - h^{\lambda\kappa}\, R^{(1)}_{\lambda\mu\nu\kappa}$$

$$R_{\lambda\mu\nu\kappa} = \frac{1}{2}\left(\partial_\mu \partial_\nu g_{\lambda\kappa} + \partial_\lambda \partial_\kappa g_{\mu\nu} - \partial_\lambda \partial_\nu g_{\mu\kappa} - \partial_\mu \partial_\kappa g_{\lambda\nu}\right)$$

$$+ g_{\rho\sigma}\left(\Gamma_{\kappa\lambda}{}^{\rho}\, \Gamma_{\mu\nu}{}^{\sigma} - \Gamma_{\nu\lambda}{}^{\rho}\, \Gamma_{\mu\kappa}{}^{\sigma}\right)$$

$$R^{(1)}_{\lambda\mu\nu\kappa} = \frac{1}{2}\left(\partial_\mu \partial_\nu h_{\lambda\kappa} + \partial_\lambda \partial_\kappa g_{\mu\nu} - \partial_\lambda \partial_\nu h_{\mu\kappa} - \partial_\mu \partial_\kappa h_{\lambda\nu}\right)$$

$$R^{(2)}_{\lambda\mu\nu\kappa} = \xi_{\rho\sigma}\left(\Gamma^{(1)}_{\kappa\lambda}{}^{\rho}\, \Gamma^{(1)}_{\mu\nu}{}^{\sigma} - \Gamma^{(1)}_{\nu\lambda}{}^{\rho}\, \Gamma_{\mu\kappa}{}^{\sigma}\right)$$

$$2\partial_\kappa h^{\kappa}{}_{\rho} - \partial_\rho h_\nu{}^{\nu} = 0$$

$$R^{(2,1)}_{\mu\nu} = h^{\lambda\kappa}\, R^{(1)}_{\lambda\mu\nu\kappa} \sim h\,\partial\partial h$$

$$0$$

$$R^{(2,2)}_{\mu\nu} = \boxed{\frac{1}{4}\, \xi^{\lambda\kappa}\, \xi^{\rho\sigma}\, \boxed{\left(\partial_\kappa h_{\lambda\rho} + \partial_\lambda h_{\rho\kappa} - \partial_\rho h_{\kappa\lambda}\right)} \left(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}\right)}$$

$$- \frac{1}{4}\, \xi^{\lambda\kappa}\, \xi^{\rho\sigma}\left(\partial_\nu h_{\lambda\rho} + \partial_\lambda h_{\rho\nu} - \partial_\rho h_{\nu\lambda}\right)\left(\partial_\mu h_{\kappa\sigma} + \partial_\kappa h_{\sigma\mu} - \partial_\sigma h_{\mu\kappa}\right)$$

$$\sim (\partial h)^2$$

→ vanishes for Hilbert gauge

Evaluate both contributions for planar wave

$$h_{\mu\nu} = e_{\mu\nu} e^{-ik_\lambda x^\lambda} + e^*_{\mu\nu} e^{+ik_\lambda x^\lambda}$$

$$R^{(2,1)}_{\mu\nu} = +\frac{1}{2}\left(e^{\lambda\kappa} e^{-ikx} + e^{\lambda\kappa *} e^{+ikx}\right)\left[(-ik_\mu)(-ik_\nu)\overset{e_{\lambda\kappa}}{e^{-ikx}} + (ik_\mu)(ik_\nu)e^*_{\lambda\kappa} e^{+ikx} + 6 \text{ other terms}\right]$$

1. Case: $e^{-ikx} e^{+ikx} = 1$ time independent

2. Case: $e^{\pm 2ikx}$: oscillating $\Rightarrow$ vanish in time average

$$R^{(2,1)}_{\mu\nu} = \frac{1}{2}\mathrm{Re}\left\{\left[k_\mu k_\nu e_{\lambda\kappa} + k_\lambda k_\kappa e_{\mu\nu} - k_\kappa k_\nu e_{\mu\kappa} - k_\mu k_\kappa e_{\lambda\nu}\right] e^{\lambda\kappa *}\right\}$$

apply Hilbert gauge

$$= \cancel{\mathrm{Re}}\left\{ k_\mu k_\nu e^{\lambda\kappa *} e_{\lambda\kappa} + \cancel{\frac{1}{2} e_{\mu\nu} k_\lambda \frac{1}{2} e^\kappa{}_\kappa k^\lambda} - \frac{1}{2}|e_\lambda{}^\lambda|^2 k_\mu k_\nu \right\}$$

$$\cancel{k^\lambda k_\lambda = \frac{\omega^2}{c^2} - \vec{k}^2 = 0}$$

$$\Rightarrow R^{(2)}_{\mu\nu} = R^{(2,1)}_{\mu\nu} + R^{(2,2)}_{\mu\nu} = \frac{1}{2} k_\mu k_\nu \left[e^{\lambda\kappa *} e_{\lambda\kappa} - \frac{1}{2}|e_\lambda{}^\lambda|^2 \right]$$

$$t^{grav}_{\mu\nu} = \frac{c^4}{16\pi G} k_\mu k_\nu \left[\underbrace{e^{\lambda\kappa *} e_{\lambda\kappa}}_{2\left[|e_{11}|^2 + |e_{12}|^2\right]} - \frac{1}{2}\underbrace{|e^\lambda{}_\lambda|^2}_{=0} \right]$$

$$(e_{\mu\nu}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e_{11} & e_{12} & 0 \\ 0 & e_{12} & -e_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \leftarrow$$

- energy-momentum tensor: energy current density
  → radiated power, depends on angles
- integrate over angles: total radiated power