4 Star Models:

$\rightarrow$ statics and dynamics of stars

4.1 Equilibrium of a star:

star = big aggregation of matter, hold together gravity

matter pressure compensates — compresses matter

$\Rightarrow$ $P_{grav} = P_{mat}$ $\hat{=}$ star equilibrium

startypes: 1) sun 2) white dwarf 3) neutron star

$\longrightarrow$ increasing strength of gravity

$\Rightarrow$ $\frac{\text{Schwarzschild radius}}{\text{star radius}}$ - - - - - - - - - - - - $\rightarrow$

Note: $M \approx M_\odot$

One needs a certain pressure / temperature in order to ignite fusion

- Sun: hydrogen atoms
  kinetic energy of the plasma leads to pressure compensating gravity pressure
- Nuclear fusion: hydrogen $\rightarrow$ helium $\rightarrow$ ...... iron
- white dwarf: helium $\rightarrow$ electron Fermi pressure comp. - grav.

• neutron star: $e + p \longrightarrow n + \nu_e$ $\Rightarrow$ very big nucleus

neutron Fermi pressure comp. - grav.

Aim: determine order of magnitude of possible masses, i.e. possible sizes of the stars

4.1.1 Pressure Distribution:

equilibrium: $P_{grav} = P_{mat} = P$

two derivation how to determine $P(r)$

1) Hydrodynamics: Euler equation

$$\rho \left\{ \underbrace{\frac{\partial \vec{v}}{\partial t} + \cancel{(\vec{v} \cdot \vec{\nabla}) \vec{v}}}_{\text{~~transport derivative~~}} \right\} = \underbrace{-\vec{\nabla} P + \underbrace{\vec{f}_0}_{\text{gravitational force per volume}}}$$

a) statics: $\vec{v} = \vec{0}$ $\qquad \vec{\nabla} P = \vec{f}_0$

gravitational force: $d\vec{F} = -dm\, \vec{\nabla} \underbrace{\Phi}_{\text{Newton potential}}$

$$\vec{f}_0 = \frac{d\vec{F}}{dV} = -\rho\, \vec{\nabla}\, \Phi \Rightarrow \vec{\nabla} P = -\rho\, \vec{\nabla}\, \Phi$$

b) isotropy: $\dfrac{dP(r)}{dr} = -\underline{\rho(r)} \dfrac{d\Phi(r)}{dr}$ $\quad \hat{=} \quad \rho^2$

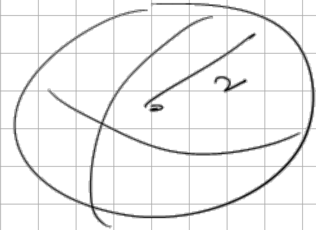
Field equation of Newtonian gravity:

$$\Delta \Phi = \frac{1}{r^2} \frac{d}{dr} \left[r^2 \frac{d\Phi(r)}{dr} \right] = 4\pi G \rho(r) \;\hat{=}\; \text{Poissonian-like equation}$$

Integrate this one:

$$\frac{d\Phi(r)}{dr} = \frac{4\pi G}{r^2}\int_0^r dr'\, r'^2\, \rho(r')$$

$$= \frac{G}{r^2}\, M(r)$$



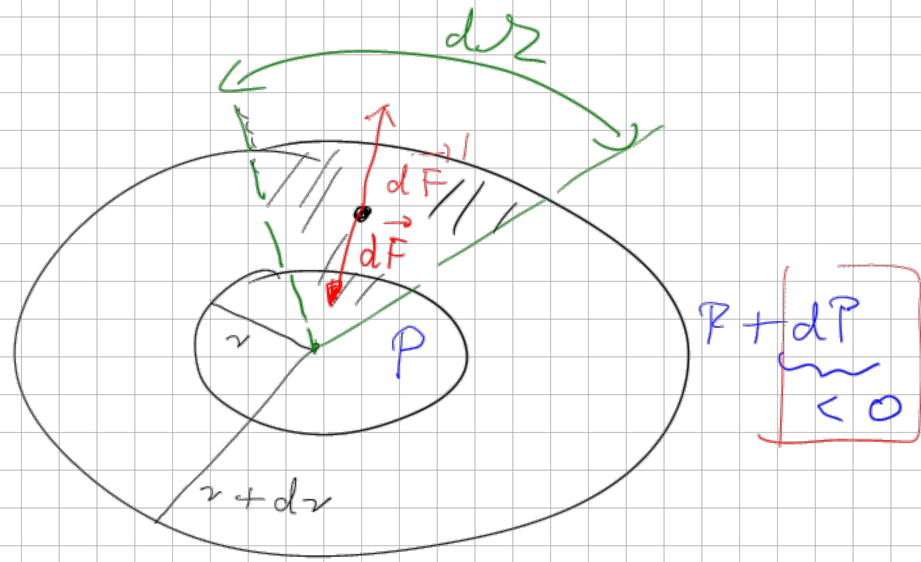
$$M(r) = 4\pi \int_0^r dr'\, r'^2\, \rho(r')$$

Equilibrium distribution for Newtonian gravity:

$$\frac{dP(r)}{dr} = -\frac{G}{r^2}\, M(r)\, \rho(r)$$

This has to be relativistically generalized

2) Based on equilibrium of forces:



$$dm = \rho\, r^2 dr\, d\Omega$$

$$d\vec{F} = -\frac{G\, M(r)}{r^2}\, dm\, \vec{e}_r$$

$\rightarrow$ gravity force

pressure force:

$$d\vec{F}' = -dP\, r^2\, d\Omega\, \vec{e}_r$$

equilibrium:

$$d\vec{F} + d\vec{F}' = \vec{0} \quad \Rightarrow \quad dP\, \cancel{r^2} = -\frac{GM(r)}{r^2}\, \rho\, \cancel{r^2}\, dr$$

$$\Rightarrow \frac{dP(r)}{dr} = -\frac{GM(r)}{r^2}\, \rho(r) \quad \checkmark \qquad \Leftarrow$$

two unknown but only one equation
We need in addition equation of state: $P = P(\rho)$
Possible equations of state:
- incompressible matter (as e.g. liquid): $\rho = \text{const.}$
- ideal gas: $P \sim \rho$
- polytrope equation of state: $P \sim \rho^{\gamma}$

4.1.2 Incompressible matter:

$$\rho(r) = \rho_0 = \text{const.} \qquad 0 \leq r \leq \underbrace{R}_{\text{star radius}}$$

$$M(r) = 4\pi \int_0^r dr' \, r'^2 \underbrace{\rho(r')}_{=\rho_0} = \frac{4\pi}{3} \rho_0 r^3$$

$$\frac{dP(r)}{dr} = -\frac{G}{r^2} \frac{4\pi}{3} \rho_0 r^3 = -\frac{4\pi G \rho_0^2}{3} r \qquad \sim \rho_0^2$$

$$P(r) = \underbrace{P_0}_{=P(r=0)} - \frac{2\pi \rho_0^2 G}{3} r^2$$

$$M = \frac{4\pi}{3} R^3 \rho_0 \qquad r_s = \frac{2GM}{c^2}$$

$r > R$: $\rho(r) = 0 \Rightarrow P(r) = 0$

$$P(R) = P_0 - \frac{2\pi \rho_0^2 G}{3} R^2 \stackrel{!}{=} 0 \Rightarrow P_0 = \frac{2\pi \rho_0^2 G}{3} R^2 = \frac{M \rho_0 G}{2R} = \rho_0 c^2 \frac{r_s}{4R}$$

$$\Rightarrow \frac{r_s}{4R} = \frac{P_0}{\rho_0 c^2}$$

- assuming incompressibility is an approximation but allows some physical insight into star properties
- $\frac{r_S}{R} = 4 \frac{P_0}{\rho_0 c^2}$ is a dimensionless measure of strength of gravity

4.1.3 Sun:

due to the high temperature within the sun one can assume an ideal gas model:

$$PV = \nu R T \overset{R = N_A k_B}{=\!=} (\nu N_A) k_B T \overset{N = \nu N_A}{=\!=} N k_B T$$

mass: $M = N m$

$$P = \frac{N k_B T}{V} = \frac{N m}{V} \frac{k_B T}{m} = \frac{M}{V} \frac{k_B T}{m} = \frac{\overset{\text{mass density}}{\rho} k_B T}{m}$$

$$\frac{r_S}{4R} \sim \frac{P}{\rho c^2} \underset{\text{ideal gas}}{=} \frac{k_B T}{m c^2}$$

hydrogen atoms: $m c^2 \approx 1\ \mathrm{GeV}$

temperature in sun: $T = 15 \cdot 10^6\ \mathrm{K} \rightsquigarrow k_B T \approx \mathrm{keV}$

$$\Rightarrow \frac{r_S}{4R} \sim 10^{-6} \ll 1 \;\hat{=}$$ Newtonian gravity is sufficient

$r_{S,\odot} = 3\,km \longrightarrow R = 750.000\,km \Rightarrow$ this not bad as an estimate

$R_\odot = 700.000\,km$

4.1.4 White Dwarfs:

Consider here a star of helium atoms.
Gravity is so strong here that atomic shell gets crushed.
$\rightarrow$ plasma: positive charges as background, electrons forces do not play a role $\rightarrow$ free electrons
$\longrightarrow$ Fermi pressure (statistical interaction)
volume V and N electrons: V/N volume per electron
Heisenberg uncertainty:

$$p \sim \frac{\hbar}{(V/N)^{1/3}} \qquad (*)$$

Dispersion:

$$\varepsilon = \sqrt{m_e^2 c^4 + p^2 c^2} = \begin{cases} m_e c^2 + \frac{p^2}{2m_e} + \dots & p \ll m_e c \quad \text{nonrelativistic} \\ pc + \dots & p \gg m_e c \quad \text{relativistic} \end{cases}$$

m_e: electron mass

Kinetic energy of matter:

$$V = \frac{4\pi}{3} R^3 \qquad (**)$$

$$E_{mat} \approx \begin{cases} N \dfrac{p^2}{2m_e} \overset{(*)}{\underset{(**)}{\approx}} \dfrac{N^{5/3} \; \hbar^2/m_e}{R^2} \sim R^{-2} & p \ll m_e c \\ pc \overset{(*)}{\underset{(**)}{\approx}} \dfrac{N^{4/3} \; \hbar c}{R} \sim R^{-1} & p \gg m_e c \end{cases}$$

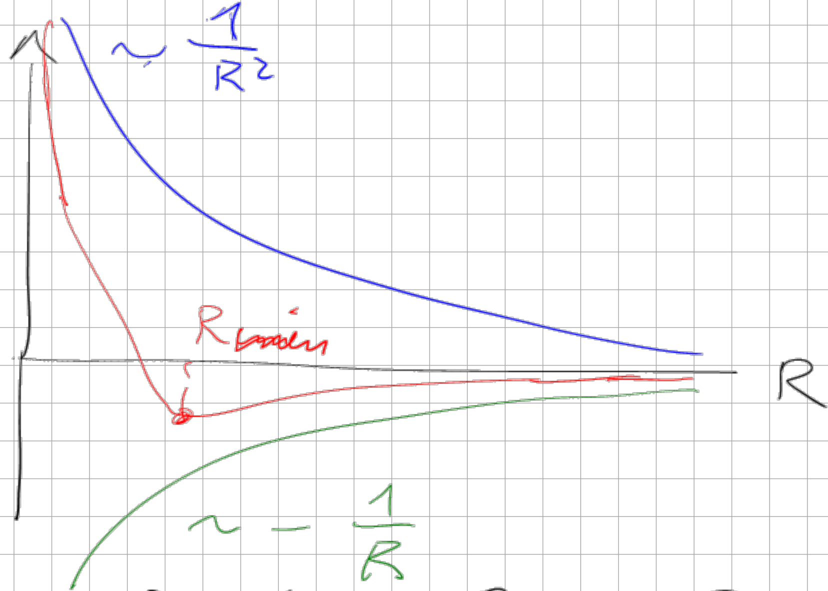
Note: all prefactors are dropped

In the following at first R-dependence of importance:

$$E_{grav} \sim - \frac{GM^2}{R} \quad , \quad M = 2 \underbrace{N}_{\text{number of electrons}} m_n$$

$$E(R) = E_{mat}(R) + E_{grav}(R)$$

$\longrightarrow$ minimum



1) Non-relativistic case

$$R_{min} \sim \frac{1}{M^2}$$

The larger the mass, the small the star radius $\longrightarrow$ Instead of nonrelativistic estimate we need to use the relativistic.

2) Relativistic case:

$$E_{mat} \sim \frac{1}{R} \quad , \quad E_{grav.} \sim \frac{-1}{R}$$

In order to avoid relativistic collapse:

$E_{mat} > E_{grav.} \Rightarrow N^{4/3} \hbar c > G M^2$, $M = 2N \cdot m_n$

$\Rightarrow$ mass condition: $M < m_n \left(\dfrac{\hbar c}{G m_n^2} \right)^{\frac{3}{2}} = M_{Ch}$

Chandrasekhar

$\Rightarrow M_C = 1.86 \, M_\odot$

Correct this concerning prefactors: $M_C = 1.2 - 1.3 \, M_\odot$