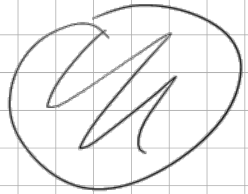
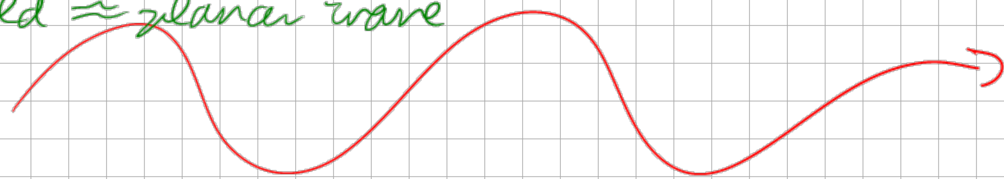
Status:

generation of gravitational waves



propagation of gravitational wave

spherical wave in far field ≈ planar wave



$$T_{\mu\nu}(\vec{x},t) = T_{\mu\nu}(\vec{x})\, e^{-i\omega t} + c.c.$$

$$\begin{cases} \neq 0 & ; \; r = |\vec{x}| \leq R_0 \\ = 0 & ; \; r > R_0 \end{cases}$$

$$h_{\mu\nu}(x) = e_{\mu\nu}\, e^{-ik_\lambda x^\lambda} + c.c.$$

$$(k^\lambda) = \begin{pmatrix} \omega/c \\ k \underbrace{\vec{x}/r}_{\text{angle dependent}} \end{pmatrix} \qquad \omega = c k$$

$$e_{\mu\nu}(\vec{k}) = -\frac{4G}{c^4 r}\left(T_{\mu\nu}(\vec{k}) - \frac{T(\vec{k})}{2}\eta_{\mu\nu}\right)$$

$$T_{\mu\nu}(\vec{k}) = \int d^3x\, T_{\mu\nu}(\vec{x})\, e^{-i\vec{k}\cdot\vec{x}}$$

$$t^{grav}_{\mu\nu} = \frac{c^4}{16\pi G} k_\mu k_\nu \left(e^{\lambda\kappa *} e_{\lambda\kappa} - \frac{1}{2}|e^\lambda{}_\lambda|^2\right)$$

3.3.4 Radiated Power per Solid Angle:

local conservation law:

$$\partial^\mu t_{\mu\nu} = 0$$

$$\nu = 0: \quad \frac{1}{c}\frac{\partial}{\partial t} t_{00} - \partial_i t_{i0} \Rightarrow \underbrace{\frac{\partial}{\partial t}\int d^3x\, \overbrace{t_{00}}^{\text{energy density}}}_{\text{conserved energy}} = -\int d^3x\, \partial_i \overbrace{\boxed{t_{i0}\, c}}^{\text{energy current density}} \underset{\text{Gauss}}{=} 0$$

$$dP = -c\, t^{i0} \cdot \underbrace{df^i}_{= \frac{x^i}{r} r^2 \underbrace{d\Omega}_{\text{inf. solid angle}}}$$

$$= -c \cdot \frac{c^4}{16\pi G} \underbrace{(-k_i)}_{= -\frac{\omega}{c}\cancel{\frac{x^i}{r}}} \underbrace{k_0}_{\omega/c} \left(e^{\lambda\kappa *} e_{\lambda\kappa} - \frac{1}{2} |e^\lambda{}_\lambda|^2 \right) \cancel{\frac{x^i}{r}} r^2 d\Omega$$

$$\frac{dP}{d\Omega} = \frac{c^3 \omega^3}{16\pi G} r^2 \left(e^{\lambda\kappa *} e_{\lambda\kappa} - \frac{1}{2} |e^\lambda{}_\lambda|^2 \right)$$

$$= \frac{c^3 \omega^2 r^2}{16\pi G} \frac{16 G^2}{c^8 r^2} \left\{ \underbrace{\left(T^{\lambda\kappa *} - \frac{T^*}{2} \eta^{\lambda\kappa}\right)\left(T_{\lambda\kappa} - \frac{T}{2} \eta_{\lambda\kappa}\right) - \frac{1}{2} \left|T^\lambda{}_\lambda - \frac{T}{2} \eta^\lambda{}_\lambda\right|^2}_{} \right\}$$

$$= T^{\lambda\kappa *} T_{\lambda\kappa} - \frac{T^*}{2} T - \frac{T}{2} T^* + \frac{T^*}{2} \frac{T}{2} \underbrace{\eta^\lambda{}_\lambda}_{=4} - \frac{1}{2} \left| T - \frac{T}{2} \underbrace{\eta^\lambda{}_\lambda}_{=4} \right|^2$$

$$\frac{dP}{d\Omega} = \frac{G \omega^2}{\pi c^5} \left\{ T^{\lambda\kappa *}(\vec{k}) T_{\lambda\kappa}(\vec{k}) - \frac{1}{2} T^*(\vec{k}) \underbrace{T(\vec{k})}_{T^\lambda{}_\lambda(\vec{k})} \right\}$$

3.3.5 Reduction to Spatial Components:

$$T^{\mu\nu}(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3} T^{\mu\nu}(\vec{k})\, e^{-i\underbrace{(\omega t - \vec{k}\vec{x})}_{= k_\lambda x^\lambda}}$$

$$D_\mu T^{\mu\nu}(x^\lambda) = 0 \xrightarrow[\text{fields}]{\text{weak}} \partial_\mu T^{\mu\nu}(x^\lambda) = 0$$

$\xrightarrow{\text{Fourier}}$ $k_\mu T^{\mu\nu}(\vec{k}) = 0$

$\nu = 0: \quad k_0 T^{00} = -k_j T^{j0}$ (1)

$\nu = i: \quad k_0 T^{i0} = -k_j T^{ji}$ (2)

unit wave vector $\hat{k}_i = \frac{k_i}{k_0}$: contains angle dependences (ϑ, φ)

(2) $T^{i0} = T^{0i} = -\hat{k}_j T^{ij}$

(1) $T^{00} = -\hat{k}_i T^{i0} = \hat{k}_i \hat{k}_j T^{ij}$

1) $T^{\mu\nu *} T_{\mu\nu} = \eta_{\mu\lambda}\eta_{\nu\kappa} T^{\mu\nu *} T^{\lambda\kappa}$

$= T^{00*}T^{00} - 2T^{0i*}T^{0i} + T^{ij*}T^{ij}$

$= \left(\hat{k}_i\hat{k}_j\hat{k}_m\hat{k}_n - 2\delta_{im}\hat{k}_j\hat{k}_m + \delta_{im}\delta_{jn}\right) T^{ij*}T^{mn}$

2) $T = T^\lambda{}_\lambda = \eta_{\lambda\kappa} T^{\lambda\kappa} = T^{00} - T^{ii} = \left(\hat{k}_i\hat{k}_j - \delta_{ij}\right) T^{ij}$

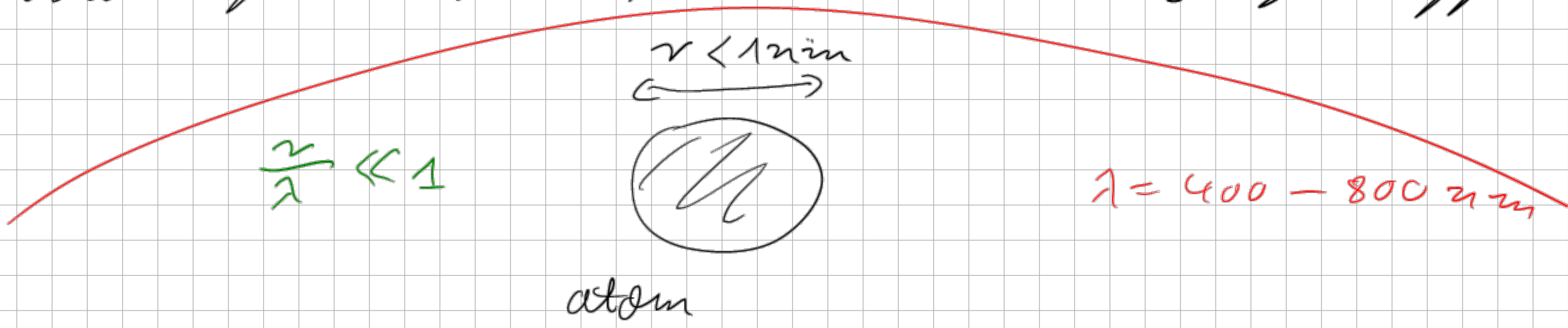
3) $T^* T = \left(\hat{k}_i\hat{k}_j\hat{k}_m\hat{k}_n - \delta_{ij}\hat{k}_m\hat{k}_n - \hat{k}_i\hat{k}_j\delta_{mn} + \delta_{ij}\delta_{mn}\right) T^{ij*}T^{mn}$

$$\frac{dP}{d\Omega} = \frac{G\omega^2}{\pi c^5} \underbrace{\Lambda_{ij,mn}}_{\text{angle dependences}} T^{ij*}(\vec{k})\, T^{mn}(\vec{k})$$

$$\Lambda_{ij,mn} = \frac{1}{2}\hat{k}_i\hat{k}_j\hat{k}_m\hat{k}_n + \frac{1}{2}\delta_{ij}\hat{k}_m\hat{k}_n + \frac{1}{2}\delta_{mn}\hat{k}_i\hat{k}_j - 2\,\delta_{im}\hat{k}_j\hat{k}_n$$
$$+\,\delta_{im}\,\delta_{jm} - \frac{1}{2}\delta_{ij}\,\delta_{mn}$$

3.3 6 Long-Wave Length Approximation:

In quantum optics this runs under the name of dipole approximation



Electromagnetic wave couples only to the electric dipole moment of the atom.

back to gravity:

$$|\vec{k}\cdot\vec{x}| \leq k\,r = \frac{2\pi}{\lambda}\,r \ll 1$$

$$T^{ij}(\vec{k}) = \int d^3x\, T^{ij}(\vec{x})\, \underbrace{e^{-i\underbrace{\vec{k}\cdot\vec{x}}_{\text{small}}}}_{=\,1 - i\vec{k}\cdot\vec{x} + \dots\ \approx\ 1} = \int d^3x\, T^{ij}(\vec{x})$$

no longer $\vec{k}$-dependent

$$T^{ij}(\vec{k}) = \frac{1}{2}\int d^3x\, x^i x^j \boxed{\partial_m \partial_n \quad T^{mn}(\vec{x})}$$

$$\partial_\mu T^{\mu\nu} = 0 \quad \Rightarrow \begin{cases} \partial_0 T^{00} + \partial_j T^{j0} = 0 \\ \partial_0 T^{0i} + \partial_j T^{ji} = 0 \end{cases} \quad \Rightarrow \quad \partial_i \partial_j T^{ij} = \underbrace{\partial_0^2}_{= -\frac{\omega^2}{c^2}} T^{00}$$

$$\Rightarrow \quad T^{ij}(\vec{k}) = -\frac{\omega^2}{2\cancel{c^2}} \int d^3x\, x^i x^j \underbrace{T^{00}(\vec{x})}$$

$$T^{00}(\vec{x}) = \frac{\rho(\vec{x})\, \cancel{c^2}}{\cancel{1 - v^2/c^2}} \quad (\text{special relativity})$$

$$v = \frac{r}{T} = \frac{1}{2\pi}\omega r = \frac{c}{2\pi} k r = \frac{c}{2\pi}\frac{2\pi}{\lambda} r = c\,\frac{r}{\lambda} \ll c$$

$\Rightarrow$ velocity contribution is negligible within long-wavelength approximation

$$\Rightarrow \quad T^{ij}(\vec{k}) = -\frac{\omega^2}{2} Q^{ij}, \qquad Q^{ij} = \int d^3x\, x^i x^j \rho(\vec{x})$$

$$\Rightarrow \frac{dP}{d\Omega} = \frac{G\omega^6}{4\pi c^5} \Lambda_{ij,mn} \quad Q^{ij*} Q^{mn}$$

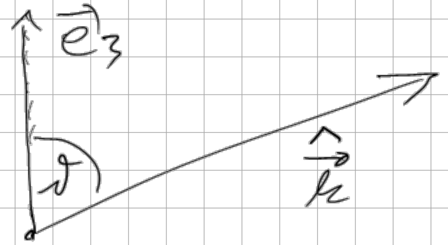
3.3.7 Radiated Power:

$$P = \int \frac{dP}{d\Omega} d\Omega = \frac{G\omega^6}{4\pi c^5} Q^{ij*} Q^{mn} \quad \int d\Omega\, \Lambda_{ij,mn}$$

Question: moments

$$\int d\Omega\, 1 = 4\pi, \quad \int d\Omega\, \hat r_i \hat r_j = ?, \quad \int d\Omega\, \hat r_i \hat r_j \hat r_m \hat r_n = ?$$

$$\hat r_i = \frac{r_i}{r_0} = \begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix}$$



Method: generating function

$$f(\vec x) = \int d\Omega\, e^{-x_m x_n \hat r_m \hat r_n} = \int d\Omega\, e^{-(\vec x\cdot\hat{\vec r})^2} = \int d\Omega\, e^{-x^2\cos^2\vartheta}$$

$$= 2\pi \int_0^\pi d\vartheta\, \sin\vartheta\, e^{-x^2\cos^2\vartheta} \underset{u=\cos\vartheta}{=} 2\pi\cdot 2\int_0^1 du\, \underbrace{e^{-x^2u^2}}_{=\sum_{n=0}^\infty \frac{(-1)^n}{n!} x^{2n}u^{2n}}$$

$x_i x_i x_j x_j$

$$\Rightarrow \boxed{f(\vec x)} = 4\pi\sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)\,n!} x^{2n} = 4\pi\left\{1 - \frac{1}{3}x^2 + \frac{1}{10}x^4 + \dots\right\}$$

$$f(\vec x = \vec 0) = 4\pi \checkmark$$

$$-\frac{\partial^2 f(\vec x)}{\partial x_i x_j}\Big|_{\vec x=\vec 0} = -\frac{\partial}{\partial x_i}\int d\Omega\,\left(-2\hat r_j \hat r_n x_n\right) e^{-x_{m'}x_{n'}\hat r_{m'}\hat r_{n'}}\Big|_{\vec x=\vec 0}$$

$$= \int d\Omega\left\{2\hat r_i\hat r_j - 4\hat r_i\hat r_j\hat r_m\hat r_n x_m x_n\right\} e^{-\dots}\Big|_{\vec x=\vec 0}$$

$$= 2\int d\Omega\, \hat r_i\hat r_j = \frac{4\pi}{3}2\delta_{ij} \Rightarrow \int d\Omega\, \hat r_i\hat r_j = \frac{4\pi}{3}\delta_{ij}$$

$$\frac{\partial^4 f(\vec{x})}{\partial x_i \partial x_j \partial x_m \partial x_n}\bigg|_{\vec{x}=\vec{0}} = \int d\Omega \left\{ \underbrace{(-2\hat{r}_j\hat{r}_n)(-2\hat{r}_i\hat{r}_m) + 4\cdot 2\,\hat{r}_i\hat{r}_j\hat{r}_m\hat{r}_n}_{=12\,\hat{r}_i\hat{r}_j\hat{r}_m\hat{r}_n} \right\}$$

$$= 8\left\{ \delta_{ij}\delta_{mn} + \delta_{im}\delta_{jn} + \delta_{in}\delta_{jm} \right\}$$

$$\int d\Omega\, \hat{r}_i\hat{r}_j\hat{r}_m\hat{r}_n = \frac{4\pi}{15}\left(\qquad \right)$$

$$\Rightarrow P = \frac{G\omega^6}{4\pi c^5}\frac{2\pi}{15}\, Q^{ij*}Q^{mn}\left(-4\,\delta_{ij}\delta_{mn} + 11\,\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}\right)$$

$$= \frac{2G\omega^6}{5c^5}\left(Q^{ij*}Q^{ij} - \frac{1}{3}Q^{ii*}Q^{jj} \right)$$

Compare with dipole radiation formula for electromagnetic waves

$$P = \frac{\omega^4}{12\pi\epsilon_0 c^4}\, p_i^* p_i$$

3.3.8 Amplitude:

We look for an amplitude of a planar gravitational wave, which has the same power P as the averaged one just calculated, propagates in x^3-direction:

$$(k^\lambda) = \frac{\omega}{c}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \quad (e^\lambda{}_\mu) = \begin{pmatrix}0&0&0&0\\0&e_{11}&e_{12}&0\\0&e_{12}&-e_{11}&0\\0&0&0&0\end{pmatrix}$$

$$t^{\mu\nu}_{grav} = \frac{c^4}{8\pi G}\, k^\mu k^\nu \left(|e_{11}|^2 + |e_{12}|^2\right)$$

energy current density per polarization:

$$\Phi = c\, t^{03} = \frac{c^3\omega^2 h^2}{8\pi G} \qquad (e_{11} = h,\ e_{12} = 0)$$

At distance D this amounts to the same

$$P = \Phi\, 4\pi D^2 = \frac{c^3\omega^2 h^2 D^2}{2G} \Rightarrow h = \sqrt{\frac{2GP}{c^3\omega^2 D^2}}$$

3.3.9 Rotating Rigid Body:

rest frame: $$(I'_{ij}) = \begin{pmatrix}I_1&0&0\\0&I_2&0\\0&0&I_3\end{pmatrix}$$

lab frame: $$I(t) = \underbrace{\alpha_{(t)}}_{} I'\, \alpha^T_{(t)} = \text{const.} + Q_{ij}\, e^{-2i\Omega t} + c.c.$$

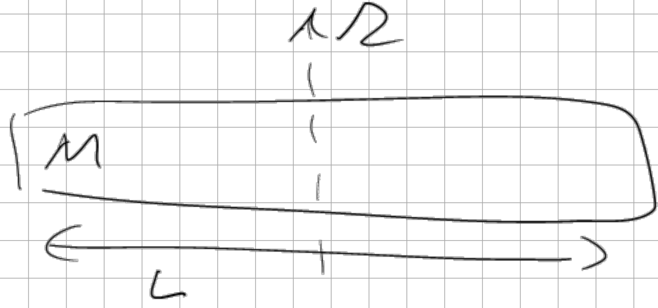
$$= \begin{pmatrix}\cos\Omega t & -\sin\Omega t & 0\\ \sin\Omega t & \cos\Omega t & 0\\ 0&0&1\end{pmatrix}, \qquad Q_{ij} = \frac{I_1 - I_2}{4}\begin{pmatrix}1&-i&0\\-i&-1&0\\0&0&0\end{pmatrix}$$

radiated power:

$$P = \frac{32\, G\, \Omega^6}{5\, c^5} (I_1 - I_2)^2$$

$\rightsquigarrow$ small numbers

3.3.10 Rotating Bar



$m = 500\,t$, $L = 20\,m$, $\Omega = 30\,\frac{1}{s}$

$\rho = \frac{m}{L}$, $I_1 = \frac{M}{12} L^2$, $I_2 \approx 0$

$\Rightarrow P = 3.6 \cdot 10^{-29}\,W \Rightarrow h = 2.2\; 10^{-37}$

$\Rightarrow$ no lab experiment yields observable gravitational waves

3.3.11 Double star system:

$\rightsquigarrow$ ellipse $\approx$ circle

$$I_1 = \frac{M_1 M_2}{M_1 + M_2} L^2, \quad L: \text{distance}$$

rotation frequency: gravitational force = centrifugal force

$$\Omega^2 = G \frac{M_1 + M_2}{L^3}$$

$$\Rightarrow P = \frac{32\, G^4}{5\, c^5} \frac{M_1^2 M_2^2 (M_1 + M_2)}{L^5}$$

double star system: ε Boo

$T = 0.268\ d \Rightarrow \Omega = \frac{2\pi}{T}$

$m_1 = 1.35\ M_\odot$, $m_2 = 0.68 M_\odot$, $M_\odot = 1.99 \cdot 10^{30} kg$

$\Rightarrow L = 1.5 \cdot 10^9\ m = 5.1\ ls = \frac{1}{100}\ \underbrace{AU}$

astronomical unit
(distance sun - earth)

distance from earth: $D = 12\ PC$

nearest star to earth: Proxima Centauri = 1.2 PS

$\Rightarrow h = 6.6 \cdot 10^{-21} \approx 10^{-20} \Rightarrow$ observable!