3.1 Gravitational Plane Waves:

- linearize gravity theory
- gauge symmetry: Hilbert gauge
- vacuum: $10 - 2 \cdot 4 = 2$ transversal degrees of freedom
- helicity ± 2

3.1.1 Linearization of Einstein field theory:

$$g_{\nu\mu} = g_{\mu\nu} = \underbrace{\eta_{\mu\nu}}_{\text{Minkowski metric}} + \underbrace{h_{\mu\nu}}_{\text{deviation } (h_{\mu\nu} = h_{\nu\mu})} \quad , \quad |h_{\mu\nu}| \ll 1$$

$\longrightarrow$ systematic perturbation theory up to first order in h

$$\Gamma_{\mu\nu\varkappa} = \frac{1}{2}\left(\partial_\mu g_{\nu\varkappa} + \partial_\nu g_{\varkappa\mu} - \partial_\varkappa g_{\mu\nu}\right) = \frac{1}{2}\left(\partial_\mu h_{\nu\varkappa} + \partial_\nu h_{\varkappa\mu} - \partial_\varkappa h_{\mu\nu}\right)$$

$$\Gamma_{\mu\nu}{}^{\lambda} = g^{\lambda\varkappa}\Gamma_{\mu\nu\varkappa} = \eta^{\lambda\varkappa}\frac{1}{2}\left(\partial_\mu h_{\nu\varkappa} + \partial_\nu h_{\varkappa\mu} - \partial_\varkappa h_{\mu\nu}\right)$$

$$= \frac{1}{2}\left(\partial_\mu h_\nu{}^\lambda + \partial_\nu h^\lambda{}_\mu - \eta^{\lambda\varkappa}\partial_\varkappa h_{\mu\nu}\right)$$

$$R_{\lambda\mu\nu}{}^{\varkappa} = \partial_\lambda \Gamma_{\mu\nu}{}^{\varkappa} - \partial_\mu \Gamma_{\lambda\nu}{}^{\varkappa} + \cancel{\Gamma_{\mu\nu}{}^{\sigma}\Gamma_{\lambda\sigma}{}^{\varkappa}} - \cancel{\Gamma_{\lambda\nu}{}^{\sigma}\Gamma_{\mu\sigma}{}^{\varkappa}}$$

$$= \frac{1}{2}\eta^{\varkappa\sigma}\left(\partial_\lambda\partial_\nu h_{\sigma\mu} + \partial_\mu\partial_\sigma h_{\lambda\nu} - \partial_\lambda\partial_\sigma h_{\mu\nu} - \partial_\mu\partial_\nu h_{\sigma\lambda}\right)$$

$$R_{\lambda\mu\nu}{}^{\lambda} = \frac{1}{2}\left(\partial_\lambda\partial_\nu h^\lambda{}_\mu + \partial_\mu\partial_\lambda h^\lambda{}_\nu - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h_\lambda{}^\lambda\right)$$

$$= R_{\mu\nu}$$

quabla operator: $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta$

Einstein field equation:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

contraction:

$$R = -\frac{8\pi G}{c^4} T$$

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right)$$

Linearization of field equations:

$$\Box h_{\mu\nu} + \partial_\mu\partial_\nu h_\lambda{}^\lambda - \partial_\lambda\partial_\nu h^\lambda{}_\mu - \partial_\mu\partial_\lambda h^\lambda{}_\nu = -\frac{16\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right)$$

contraction lead to coupling between h's

3.1.2 Infinitesimal coordinate transformation:

$$x^{\cancel{\alpha}} \to x'^\mu = x^\mu + \underbrace{\varepsilon^\mu(x)}_{\text{small deviation}}$$

reciprocal 4-bein:

$$e_\alpha{}^\mu = \frac{\partial x'^\mu}{\partial x^\alpha} = \partial_\alpha\left(x^\mu + \varepsilon^\mu(x)\right) = \delta^\mu{}_\alpha + \partial_\alpha \varepsilon^\mu \qquad (1)$$

corresponding 4-bein

$$e^\alpha{}_\mu = \delta^\alpha{}_\mu - \partial_\mu \varepsilon^\alpha \qquad (3) \text{ due (1) and (2)}$$

orthonormality : $e_\alpha{}^\mu e^\alpha{}_\nu = \delta^\mu{}_\nu$ (2)

covariant metric :

$$g'_{\mu\nu} = e^\alpha{}_\mu \; e^\beta{}_\nu \; g_{\alpha\beta}$$

$$= \left[\delta^\alpha{}_\mu - \partial_\mu \varepsilon^\alpha\right]\left[\delta^\beta{}_\nu - \partial_\nu \varepsilon^\beta\right]\left[\eta_{\alpha\beta} + h_{\alpha\beta}\right]$$

→ expand ~~up~~ to first order in ε and h

$$g'_{\mu\nu} = \eta_{\mu\nu} + h^?_{\mu\nu}\left(x + \cancel{\varepsilon(x)}\right)$$

$$h'_{\mu\nu}(x) = h_{\mu\nu}(x) \underbrace{- \partial_\mu \varepsilon_\nu(x) - \partial_\nu \varepsilon_\mu(x)}_{4 \text{ gauge functions}}$$

3.1.3 Hilbert Gauge:

Aim: decouple equations for h's by choosing a particular gauge

Hilbert gauge: $\partial_\nu h'_\mu{}^\mu = 2 \partial_\mu h'_\nu{}^\mu$

show later that this accomplishes, indeed, a decoupling

At first construct gauge functions such that h' fulfills Hilbert gauge

$$\partial_\nu \left(h_\mu{}^\mu - 2\partial_\mu \varepsilon^\mu - \cancel{\partial_\mu \varepsilon^\mu}\right) = 2\, \partial_\mu \left(h_\nu{}^\mu - \partial_\nu \varepsilon^\mu - \partial^\mu \varepsilon_\nu\right)$$

$$\partial_\nu h_\mu{}^\mu - 2\, \cancel{\partial_\nu \partial_\mu \varepsilon^\mu} = 2\, \partial_\mu h_\nu{}^\mu - \cancel{2\, \partial_\nu \partial_\mu \varepsilon^\mu} - 2\, \Box \varepsilon_\nu$$

Differential equation to determine the gauge function

$$\Box \varepsilon_\nu = \partial_\mu h_\nu{}^\mu - \frac{1}{2} \partial_\nu h_\mu{}^\mu$$

Inhomogeneity vanishes provided that h fulfills Hilbert gauge

$\Rightarrow$ 4 decoupled wave equations for ε:

$$\varepsilon_\nu = \varepsilon_\nu^{hom} + \underbrace{\varepsilon_\nu^{part}(\vec{x},t)}$$

$$= \int d^3x \int dt \, \underbrace{G(\vec{x},t;\vec{x}',t')}\left(\partial_\mu h_\nu{}^\mu - \frac{1}{2}\partial_\nu h_\mu{}^\mu\right)(\vec{x}',t')$$

$$= -\frac{1}{4\pi}\frac{1}{|\vec{x}-\vec{x}'|}\underbrace{\delta\left(t'-t+\frac{|\vec{x}-\vec{x}'|}{c}\right)}_{\text{retardation}}$$

Due to covariance principle of general relativity the Einstein field equation do have the same form after having performed this infinitesimal coordinate transformation

$$\Box h'_{\mu\nu} + \cancel{\partial_\mu \partial_\nu h'_\lambda{}^\lambda} - \cancel{\partial_\nu \underbrace{\partial_\lambda h'^\lambda{}_\mu}_{=\frac{1}{2}\partial_\mu h_\lambda{}^\lambda}} - \cancel{\partial_\mu \underbrace{\partial_\lambda h'^\lambda{}_\nu}_{=\frac{1}{2}\partial_\nu h_\lambda{}^\lambda}} = -\frac{16\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right)$$

Hilbert gauge

$$\Rightarrow \Box h'_{\mu\nu} = -\frac{16\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right)$$

3.1.4 Electromagnetic Waves:

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $(A^\mu) = \begin{pmatrix} \varphi/c \\ \vec{A} \end{pmatrix}$

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

inhomogeneous Maxwell equations

$$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu, \quad (j^\nu) = \begin{pmatrix} c\varrho \\ \vec{j} \end{pmatrix}$$

$$\partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu \right) = \Box A^\nu - \partial^\nu \underbrace{(\partial_\mu A^\mu)}_{\text{coupling of all four components}} = \mu_0 j^\nu$$

Use gauge symmetry: $A'_\mu = A_\mu + \partial_\mu \chi$

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = F_{\mu\nu} + \underbrace{(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\chi}_{\equiv 0} = F_{\mu\nu}$$

Lorentz gauge: $\partial_\mu A^\mu = 0$

Let us assume $\partial_\mu A^\mu \neq 0$. Can we choose a gauge function χ such that $\partial_\mu A'^\mu = 0$? YES

$$\partial_\mu (A^\mu + \partial^\mu \chi) = \partial_\mu A^\mu + \Box \chi \overset{!}{=} 0 \Rightarrow \Box \chi = -\partial_\mu A^\mu$$

$$\Rightarrow \quad \chi = \chi^{hom} + \underbrace{\chi^{part}}_{}$$

$$= -\int d^3x' \int dt\; G(\vec{x}, t; \vec{x}', t')\,(\partial_\mu A^\mu)(\vec{x}', t')$$

decoupled equations: $\Box A^\nu = \mu_0 j^\nu$

4 components of A^ν $-$ 1 $=$ 3 degrees of freedom

Vacuum: $j^\nu = 0 \Rightarrow \Box A^\nu = 0$

$$A^\mu = \underbrace{e^\mu}_{\text{polarization vector}} e^{-ik^\lambda x_\lambda} \underbrace{+\ h.c.}_{\text{added to make } A^\mu \text{ real}} \qquad \text{plane wave}$$

$$\Rightarrow \quad k^\lambda k_\lambda = \varepsilon_{\mu\nu} k^\mu k^\nu = 0 \Rightarrow \omega(\vec{k}) = c\,|\vec{k}| \qquad (k^\lambda) = \begin{pmatrix} \omega/c \\ \vec{k} \end{pmatrix}$$

Lorentz gauge: $\partial_\mu A^\mu = 0 \Rightarrow k_\mu e^\mu = 0$

Let us investigate the consequence of a gauge function χ, which solves homogeneous wave equation

$$\chi^{hom} = \delta\ e^{-ik^\lambda x_\lambda} \qquad (\delta \text{ is here arbitrary})$$

$$A'^\mu = A^\mu + \underbrace{\partial^\mu}_{= \frac{\partial}{\partial x_\mu}} \chi^{hom} = \underbrace{\left(e^\mu - i\delta k^\mu\right)}_{= e'^\mu} e^{-ik^\lambda x_\lambda}$$

radiation gauge: $A'^0 = 0 \Rightarrow e'^0 = e^0 - i\delta k^0 \stackrel{!}{=} 0$

$$\Rightarrow \delta = -i \frac{e^0}{k^0}$$

$\partial_\mu A'^\mu = 0 \Rightarrow \partial_i A'^i = 0$ Coulomb gauge

Result:

$A^0 = 0 \hat{=} \varphi = 0 \Rightarrow e^0 = 0 \Rightarrow (e^\mu) = \begin{pmatrix} 0 \\ \vec{e} \end{pmatrix}$

$\partial_i A^i = 0 \Rightarrow \vec{k} \cdot \vec{e} = 0$ transversal plane wave

Let us look at a plane wave propagating in x^3-direction

$$\vec{k} = k\,\vec{e}_3 \Rightarrow (k^\mu) = \begin{pmatrix} k \\ 0 \\ 0 \\ k \end{pmatrix}, \quad (k_\mu) = \begin{pmatrix} k \\ 0 \\ 0 \\ -k \end{pmatrix}$$

$$\vec{e}' \cdot \vec{k} = 0 \Rightarrow \vec{e}' \cdot \vec{e}_3 = 0 \Rightarrow \vec{e}' = \begin{pmatrix} e_1 \\ e_2 \\ 0 \end{pmatrix}$$

$$(A^\mu) = \begin{pmatrix} 0 \\ e_1 \\ e_2 \\ 0 \end{pmatrix} e^{i(\vec{k}\vec{x} - \omega t)}$$

Check for helicity: Rotation around x_3

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi & 0 \\ 0 & -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$e' = \Lambda e \Rightarrow \begin{pmatrix} e'^1 \\ e'^2 \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} e^1 \\ e^2 \end{pmatrix}$$

$$e_\pm = e^1 \pm i e^2 \Rightarrow e^1 = \frac{1}{2}(e_+ + e_-), \quad e^2 = \frac{-i}{2}(e_+ - e_-)$$

$$e'_\pm = e'^1 \pm i e'^2 = e^{\mp i \varphi \cdot 1} e_\pm \quad \Rightarrow \text{helicity } \mp 1$$

$$e_+ = 0: \quad (A^\mu) = \frac{e_-}{2} \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix} e^{ik(x^3 - ct)}$$

$$e_- = 0: \quad (A^\mu) = \frac{e_+}{2} \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix} e^{ik(x^3 - ct)}$$

circular polarized plane wave

3.1.5 Plane Wave (in Vacuum):

$$h_{\mu\nu}(x) = \underbrace{e_{\mu\nu}}_{\text{polarization tensor } (= e_{\nu\mu})} e^{-ik^\lambda x_\lambda} \cancel{+ \text{h.c.}}$$

Hilbert gauge: $\partial_\nu h_\mu{}^\nu = 2\, \partial_\mu h'_\nu{}^\nu \Rightarrow \underline{k_\nu}\, \underline{e_\mu{}^\nu} = 2 k^\nu e_{\nu\mu}$

wave equation $\Box h_{\mu\nu} = 0$ ← vacuum

$$\Downarrow$$

$$k^\lambda k_\lambda = 0 \Rightarrow \omega = c|\vec{k}|$$

Gravitational waves also propagate with light velocity.

$\Rightarrow$ checked within multi-messenger astronomy

Propagation in x^3-direction

$$(k^\lambda) = \begin{pmatrix} k \\ 0 \\ 0 \\ k \end{pmatrix}, \qquad (k_\lambda) = \begin{pmatrix} k \\ 0 \\ 0 \\ -k \end{pmatrix}$$

$$k_\nu \left(e_{00} - e_{11} - e_{22} - e_{33}\right) = 2k\left(e_{\nu 0} + e_{\nu 3}\right)$$

4 constraints:

$$\nu = 0: \quad k\left(e_{00} - e_{11} - e_{22} - e_{33}\right) = 2k\left(e_{00} + e_{03}\right) \qquad (1)$$

$$\nu = 1: \quad 0 = 2k\left(e_{10} + e_{13}\right) \Rightarrow e_{10} = e_{01} = -e_{13}$$

$$\nu = 2: \quad 0 = 2k\left(e_{20} + e_{23}\right) \Rightarrow e_{20} = e_{02} = -e_{23}$$

$$\nu = 3: \quad -k\left(e_{00} - e_{11} - e_{22} - e_{33}\right) = 2k\left(e_{30} + e_{33}\right) \qquad (2)$$

$$(1) + (2): \quad e_{03} = e_{30} = -\frac{e_{00} + e_{33}}{2}$$

$$(1) - (2): \quad e_{22} = -e_{11}$$

In total: $10 - 4 = 6$ degrees of freedom

$$e_{00},\ e_{11},\ e_{33},\ e_{12},\ e_{13},\ e_{23}$$

$$(e_{\mu\nu}) = \begin{pmatrix} \cancel{e_{00}} & \cancel{-e_{13}} & \cancel{-e_{23}} & \cancel{-\frac{e_{00}+e_{33}}{2}} \\ \cancel{-e_{13}} & e_{11} & e_{12} & \cancel{e_{13}} \\ \cancel{-e_{23}} & e_{12} & -e_{11} & \cancel{e_{23}} \\ \cancel{-\frac{e_{00}+e_{33}}{2}} & \cancel{e_{13}} & \cancel{e_{23}} & \cancel{e_{33}} \end{pmatrix}$$

They all can be chosen without loss of generality to zero