Fluxoid conservation:

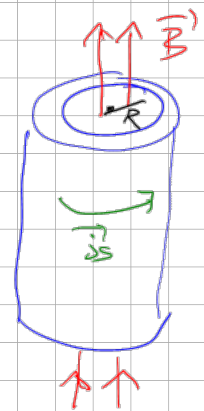
$$\frac{\partial}{\partial t}\phi' = 0 \,, \quad \phi' = \int_F \vec{B}\cdot d\vec{F} + \Lambda_s \oint_{\partial F} \vec{j}_s \cdot d\vec{r} = \frac{1}{e_s}\oint_{\partial F} \vec{p}_s \cdot d\vec{r} \,, \quad \vec{p}_s = m\vec{v}_s + e_s\vec{A}$$

Bohr-Sommerfeld quantization:

$$\underset{n\in\mathbb{N}_0}{\phi' = n\,\phi_0} \,, \quad \phi_0 = \frac{h}{e_s} \overset{e_s=2e}{=} 2.07\cdot10^{-15}\,\mathrm{Tm}^2 \quad \text{flux quantum}$$

## 4.7 Experiment: Flux Quantum

### 4.7.1 General Idea:




- superconducting current in cylinder shell
- $\vec{B}$ switched off: currents persist

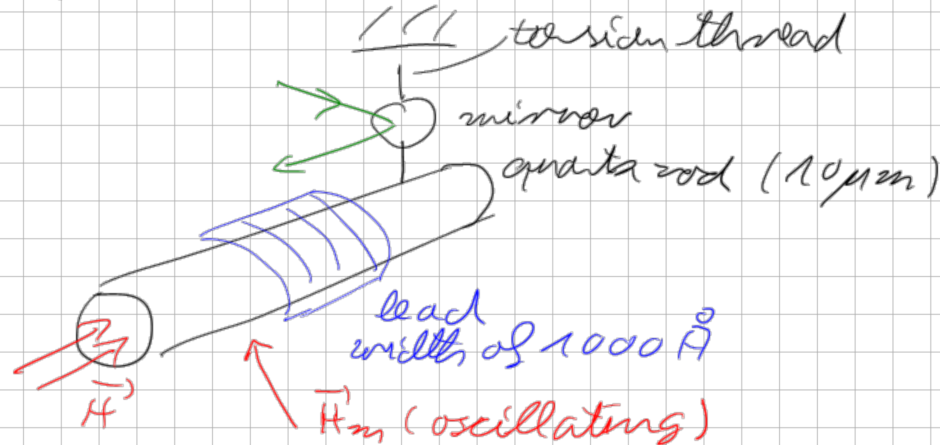
$$B = \frac{\phi_0}{\pi R^2} \overset{R=5\cdot10^{-6}\,\mathrm{m}}{=} 26\,\mu\mathrm{T} \quad (\approx B_{earth} = 30\,\mu\mathrm{T})$$

→ shield earth magnetic field

### 4.7.2 Method of Doll–Näbauer (Munich):

resonance between external frequency $\omega_m$ and oscillation frequency

→ strong elongations



1) observation: $H_1 : H_2 : H_3 : \ldots = 1 : 2 : 3 \ldots$

→ $\mu_0 H \pi R^2$ = magnetic flux is quantised

2) Deficiency: quantisation (of magnetic flux) not perfect as fluxoid is quantised, i.e. superconducting currents → ≈ 10% of fluxoid can not be measured


## 4.7.3 Method of Deaver and Fairbank (Stanford):

- also resonance
- but here: longitudinal mechanical motion, induces induction voltage in coils at end / beginning of cylinder (frequency 100 Hz)

## 4.7.4 Discussions:

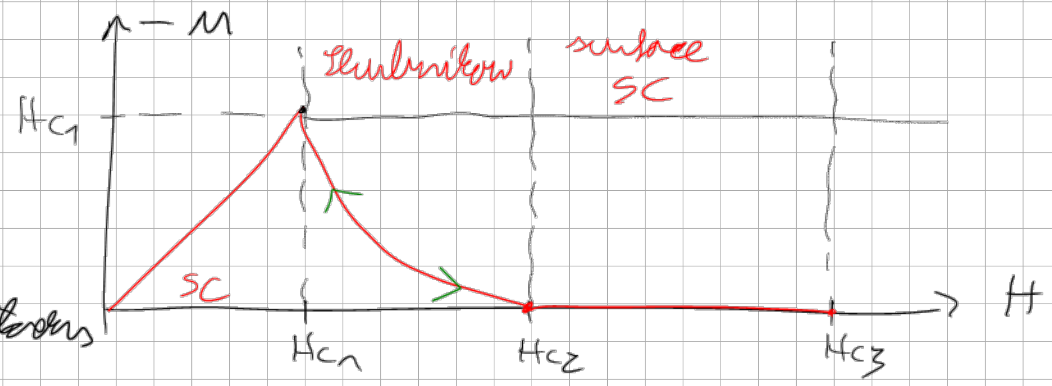
- magnet flux is connected by superconducting currents → fluxoid is quantised
- quantisation: $e_s = 2e$
- According to London theory all Cooper pairs contribute to nth flux quantum. Transition from n to n' flux quanta implies that all Cooper pairs adjust coherently

⇒ superconductivity is a macroscopic quantum phenomenon.

## 4.8 Structure of Elementary Flux Quantum:

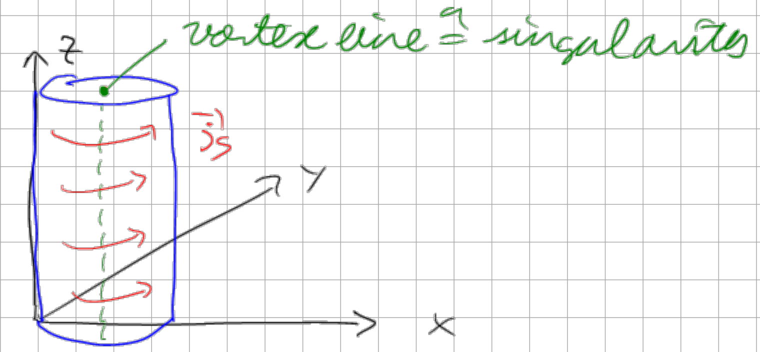
Type II superconductors

- At $H_{c1}$ the first quanta enter, number of flux quanta increase → Abrikosov vortex lattice
- At $H_{c2}$ the whole SC only consists of vortices, normal conducting electrons




## 4.8.1 Inhomogeneous Helmholtz Equation:

- inner part: normal conducting in form of this singularity
- going away from singularity: decrease of $\vec{j}_s$
- cylinder symmetry




$$\phi' = \int_F \vec{B} \cdot d\vec{F} + \underbrace{\Lambda_s}_{\mu_0 \lambda_L^2} \oint_{\partial F} \vec{j}_s \cdot d\vec{s} \overset{!}{=} \phi_0$$

$$\Rightarrow \int_F \Big(\vec{B} + \mu_0 \lambda_L^2 \,\mathrm{rot}\, \underbrace{\vec{j}_s}_{\text{steady state}}\Big)\, d\vec{F} = \phi_0 = \int_F \Big(\vec{B} + \lambda_L^2 \underbrace{\mathrm{rot}\,\mathrm{rot}\,\vec{B}}_{=-\Delta\vec{B}}\Big) \cdot d\vec{F}$$

$= \vec{j}$

Maxwell

$$\frac{1}{\mu_0}\,\mathrm{rot}\,\vec{B} - \cancel{\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}}$$

$$= \int_F \delta^{(2)}(\vec{x})\, d\vec{F} \cdot \vec{e}_z$$

valid for all areas $F$

$$\Delta \vec{B} - \frac{1}{\lambda_L^2}\vec{B} = -\frac{\phi_0}{\lambda_L^2}\,\delta^{(2)}(\vec{x})\,\vec{e}_z$$

## 4.8.2 Simplification:

cylinder symmetry

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{\lambda_L^2}\right)\vec{B}(r,\varphi,z) = -\frac{\phi_0}{\lambda_L^2}\,\delta^{(2)}(\vec{x})\,\vec{e}_z$$

$$\vec{B}(r) = B_z(r)\,\vec{e}_z$$

$$\Rightarrow \quad \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{\lambda_L^2}\right) B_z(r) = -\frac{\phi_0}{\lambda^2}\,\delta^{(2)}(\vec{x})$$

## 4.8.3 Cylinder Functions:

homogeneous part:

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \frac{1}{\lambda_L^2} - \frac{n^2}{r^2}\right) B_z(r) = 0\,, \quad n \in \mathbb{N}_0$$

dimensionless coordinate: $\quad a = \frac{r}{\lambda_L}$

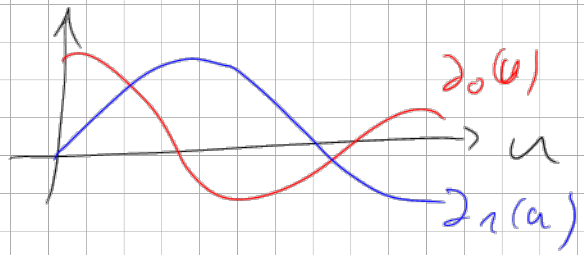
$$\left(\frac{d^2}{da^2} + \frac{1}{a}\frac{d}{da} + 1 - \frac{n^2}{a^2}\right) B_z(a) = 0\,, \; n \in \mathbb{N}_0 \qquad (*)$$

linear ordinary diff. eq. of second order: 2 fundamental solutions

| Bessel functions | Von Neumann functions |
|---|---|
| cylinder functions of 1st kind | — of 2nd kind |
| of order $n$ | of order $n$ |
| $J_n(a)$ | $Y_n(a)$ |

$$\lim_{u\to 0} J_n(u) = \delta_{n,0}$$

$$\lim_{u\to \infty} J_n(u) = 0$$




$$Y_0(u) \sim \ln u \quad , u \to 0$$

$$Y_n(u) \sim \frac{1}{u^n} \quad , u \to 0 , \quad n \in \mathbb{N}$$


linear combination: Hankel functions (cylinder functions of 3rd kind)

$$H^{\pm}_n(u) = J_n(u) \pm i\, Y_n(u)$$

## 4.8.4 Modified Cylinder Functions:

$$u = \varepsilon u' \xrightarrow{(*)} \left( \frac{d^2}{du'^2} + \frac{1}{u'} \frac{d}{du'} - 1 - \frac{n^2}{u'^2} \right) Z_n(u') \quad , n \in \mathbb{N}_0$$

2 fundamental solutions:

Modified Bessel function

$$I_n(u) = J_n(iu)\, i^{-n}$$

$$I_0(u) \Rightarrow \frac{1}{\sqrt{2\pi u}} e^{u} , \quad u \to \infty$$

$$\lim_{u\to 0} I_n(u) = \delta_{n,0}$$

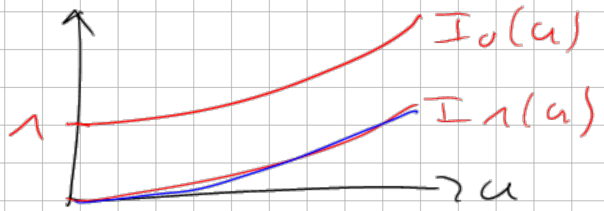
Modified Hankel function

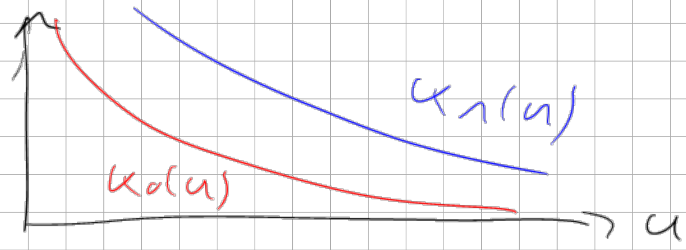
$$K_n(u) = \frac{\pi}{2}\, i^{n+1} H^{+}_n(iu)$$

$$K_n(u) \Rightarrow \sqrt{\frac{\pi}{2u}}\, e^{-u} , \quad u \to \infty$$

$$K_0(u) \Rightarrow -I_0(u) \ln\frac{u}{2} \quad u \to 0$$

$$K_1(u) \Rightarrow \frac{1}{u} \quad u \to 0$$




## 4.8.5 Solution of homogeneous Helmholtz equation

$$B_z(r) = b_1 I_0\left(\frac{r}{\lambda_L}\right) + b_2 K_0\left(\frac{r}{\lambda_L}\right) \qquad r > 0$$

boundary condition: $B_z(r) \to 0$ , $r \to \infty$ $\Rightarrow$ $b_1 = 0$, $b_2 = b$

$$B_z(r) = b\, K_0\left(\frac{r}{\lambda_L}\right)$$ is the solution of inhom. Helmholtz equation for $r>0$

$b$ has to be determined by singularity at $r = 0$

## 4.8.6 Proportionality Constant:

$$\int_F \vec{B}\cdot d\vec{F} + \mu_0 \lambda_L^2 \oint_{\partial F} \vec{j}_s \cdot d\vec{r} = \phi_0$$

Maxwell

$$= \vec{j} = \frac{1}{\mu_0} \operatorname{rot} \vec{B} , \quad \vec{B}(\vec{r}) = B_z(r)\, \vec{e}_z$$

1) $\vec{j}_s(\vec{r}) = ?$

$$\vec{j}_s(\vec{r}) = \frac{1}{\mu_0}\left\{ \left(\cancel{\frac{1}{r}\frac{\partial B_z}{\partial \varphi}}\right)\vec{e}_r + \left( - \frac{\partial B_z}{\partial r}\right)\vec{e}_\varphi + \left( \quad \right)\vec{e}_z \right\}$$

$$\vec{j}_s(\vec{r}) = j_\varphi(r)\,\vec{e}_\varphi , \quad j_\varphi(r) = -\frac{1}{\mu_0}\frac{\partial B_z(r)}{\partial r} = -\frac{b}{\mu_0 \lambda_L} \underbrace{K_0'\left(\frac{r}{\lambda_L}\right)}_{= -K_1\left(\frac{r}{\lambda_L}\right)} = \frac{b}{\mu_0 \lambda_L} K_1\left(\frac{r}{\lambda_L}\right)$$

2) $$\oint_{\partial F} \vec{j}_s \cdot d\vec{r} \underset{\substack{\uparrow \\ \text{circle of radius } R}}{=} \int_0^{2\pi} d\varphi \; j_\varphi(R)\, \vec{e}_\varphi \cdot R\,\vec{e}_\varphi = \frac{2\pi R b}{\mu_0 \lambda_L} K_1\left(\frac{R}{\lambda_L}\right)$$

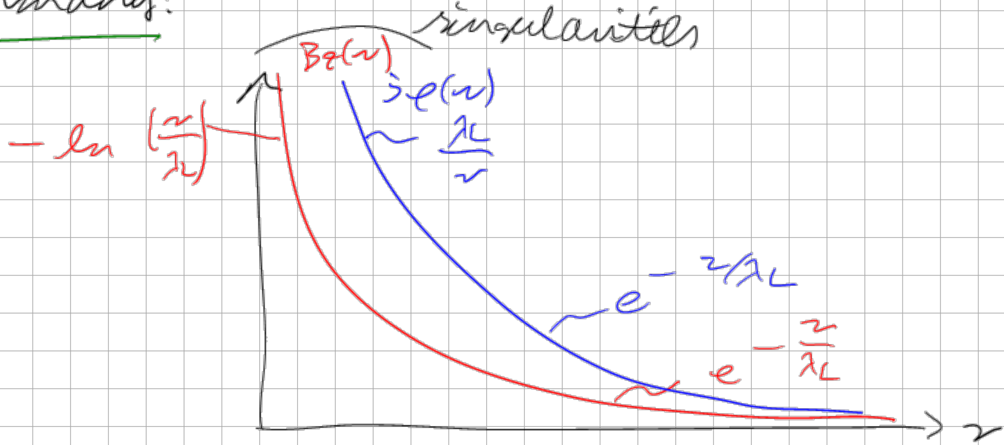
3) $$\int_F \vec{B} \cdot d\vec{r} = \int_0^{2\pi} d\varphi \int_0^R dr\, r \; B_z(r)\, \underbrace{\vec{e}_z \cdot \vec{e}_z}_{=1} = 2\pi b \lambda_L^2 \int_0^{\frac{R}{\lambda_L}} du \; \underbrace{u K_0(u)}_{= -\frac{d}{du}\left[u K_1(u)\right]}$$

$$= 2\pi b \lambda_L^2 \left\{ 1 - \frac{R}{\lambda_L} K_1\left(\frac{R}{\lambda_L}\right) \right\}$$

$$\Rightarrow \quad \phi_0 = 2\pi b \lambda_L^2 \quad \Rightarrow \quad b = \frac{\phi_0}{2\pi \lambda_L^2} \quad \Rightarrow \quad \vec{B}(r) = \underbrace{B_z(r)}_{= \frac{\phi_0}{2\pi\lambda_L^2} K_0\left(\frac{r}{\lambda_L}\right)} \cdot \vec{e}_z$$

$$\Rightarrow \quad \vec{j}_s(\vec{r}) = j_\varphi(r)\, \vec{e}_\varphi, \quad j_\varphi(r) = \frac{\phi_0}{\mu_0 2\pi \lambda_L^3} K_1\left(\frac{r}{\lambda_L}\right)$$

## 4.8.9 Summary:




- $B_z$, $j_s$ vary on length scale $\lambda_L$
- London theory leads to singularities for $B_z(r)$ and $j_s(r)$ for $r \to 0$

inconsistent as superconductivity breaks down at $B_c$

- This inconsistency arises as London theory assumes that $n_s$ is constant.

BCS theory: $n_s$ is inhomogeneous $\rightarrow$ Ginzburg – Landau theory

## 4.9 Extension of London Theory:

The assumption is that there exists a wave function $\Psi$ describing ALL superconducting electrons (Cooper pairs) $\rightarrow n_s = |\Psi|^2$ superconducting electron density