Ginzburg-Landau Theory

$$G_s = G_n + \int_V dV \left\{ \underbrace{\alpha}_{=\alpha_c'(T-T_c)} \underbrace{|\psi|^2}_{=n_s} + \overbrace{\frac{\beta}{2}}^{T\text{-independent}} |\psi|^4 + \underbrace{\frac{1}{2m_s}}_{=2m} \left| \Big(\underbrace{-i\hbar\vec{\nabla}}_{=\hat{\vec{p}}} - \underbrace{e_s}_{=2e} \vec{A} \Big) \psi \right|^2 + \frac{1}{2\mu_0} \left(\vec{B} - \mu_0 \vec{H} \right)^2 \right\} ; \vec{B} = \mathrm{rot}\,\vec{A}$$

Landau

$$\longrightarrow \alpha + \beta |\psi|^2 = 0$$

$$n_s = -\frac{\alpha}{\beta} = \underline{|\psi|^2}$$

$$\underbrace{-\alpha = \alpha_c'(T_c - T)}_{>0}$$

- Extremization: GL equations
- nonlinear (self-consistency) $\longrightarrow$ numerical solution
- in special situations: analytic approximations

5.4 Characteristic lengths:

- London penetration length: λ_L
- Coherence length: ξ

type I SC

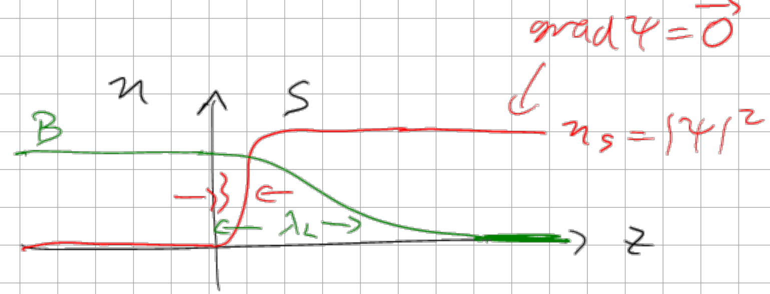
$\lambda_L \ll \xi$

type II SC

$\lambda_L \gg \xi$

5.4.1 Extreme SC type II:

$$\vec{j}_s = \cancel{\frac{\hbar}{2m_s i}\left(\psi^* \vec{\nabla}\psi - \psi \vec{\nabla}\psi^*\right)} \underbrace{- \frac{e_s^2 n_s}{m_s} \vec{A} \overbrace{|\psi|^2}^{=n_s}}_{\hat{=}\text{ London theory}}$$



London penetration length:

$$\lambda_L = \sqrt{\frac{m_s}{e_s^2 \mu_0 \underbrace{|\psi|^2}_{=n_s}}} \sim (T_c - T)^{-\frac{1}{2}} \longrightarrow \infty$$

divergence with critical exponent 1/2

5.4.2 Extreme SC of Type I:

$\vec{B} = \vec{0}, \quad \vec{H} = \vec{0}$

Ginzburg-Landau equation:

$$-\frac{\hbar^2}{2m_s}\Delta\psi + \underbrace{\alpha}_{=-\mu}\psi + \underbrace{\beta}_{=g}|\psi|^2\psi = 0$$

Gross-Pitaevskii eq.

Specialise to 1D:

$$-\frac{\hbar^2}{2m_s}\frac{d^2\psi}{dz^2} + \alpha\psi + \beta\psi^3 = 0 \quad , \quad \psi^* = \psi$$

dimensionless quantities:

1) $\bar{\psi} = \frac{\psi}{\psi_0}, \quad \psi_0 = \sqrt{-\frac{\alpha}{\beta}} \Rightarrow -\frac{\hbar^2}{2m_s}\frac{d^2\bar{\psi}}{dz^2} + \alpha\bar{\psi} - \alpha\bar{\psi}^3 = 0$

length scale: $\xi = \sqrt{-\frac{\hbar^2}{2m_s\alpha}} \qquad \alpha = \alpha_0'(T - T_c), \quad T < T_c$

$$\Rightarrow \xi^2\frac{d^2\bar{\psi}}{dz^2} + \bar{\psi} - \bar{\psi}^3 = 0$$

2) $\bar{z} = \frac{z}{\lambda_L}$, Ginzburg-Landau parameter $\varkappa = \frac{\lambda_L}{\xi} \ll 1$ type I

$$\frac{1}{\varkappa^2}\frac{d^2\bar{\psi}}{d\bar{z}^2} + \bar{\psi} - \bar{\psi}^3 = 0 \quad \Big| \cdot \frac{d\bar{\psi}}{d\bar{z}}, \quad \text{integrate}$$

$$\left.\begin{aligned} &\frac{1}{2\varkappa^2}\left(\frac{d\bar{\psi}}{d\bar{z}}\right)^2 + \frac{1}{2}\bar{\psi}^2 - \frac{1}{4}\bar{\psi}^4 + c_1 = 0 \\ &\bar{z} \to \infty: \ \bar{\psi} \to 1, \ \frac{d\bar{\psi}}{d\bar{z}} \to 0 \end{aligned}\right\} \frac{1}{2} - \frac{1}{4} + c_1 = 0 \Rightarrow c_1 = -\frac{1}{4}$$

$$\frac{1}{\kappa^2}\left(\frac{d\bar{\psi}}{d\bar{z}}\right)^2 = \frac{1}{2}\left(1-\bar{\psi}^2\right)^2$$

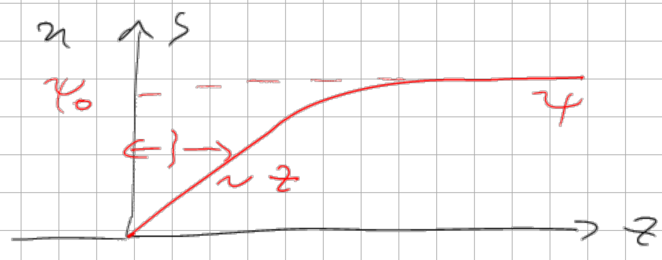
separating variables method

$$\left.\begin{aligned} &\bar{\psi}(\bar{z}) = \tanh\left(\frac{\kappa}{\sqrt{2}}\bar{z} + c_2\right); \quad \bar{z} > 0 \\ &\bar{z} \leq 0: \quad \bar{\psi} = 0 \end{aligned}\right\} \quad \text{continuity: } c_2 = 0$$

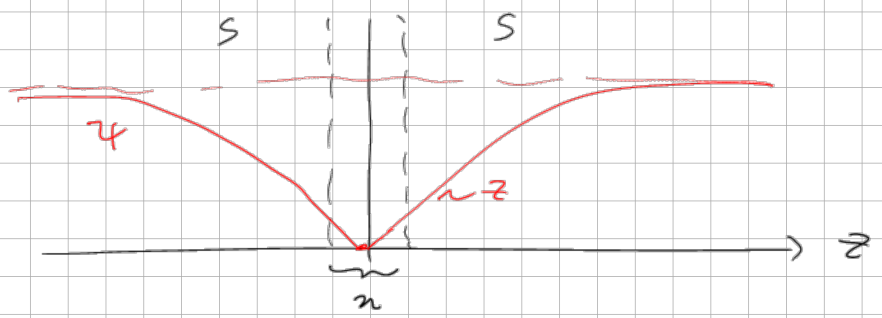
$$\Rightarrow \bar{\psi}(\bar{z}) = \begin{cases} \tanh\left(\frac{\kappa}{\sqrt{2}}\bar{z}\right) & ; \ \bar{z} \geq 0 \\ 0 & ; \ \bar{z} \leq 0 \end{cases} \Rightarrow \psi(z) = \begin{cases} \psi_0 \tanh\left(\frac{z}{\sqrt{2}\xi}\right) & ; \ z \geq 0 \\ 0 & ; \ z \leq 0 \end{cases}$$

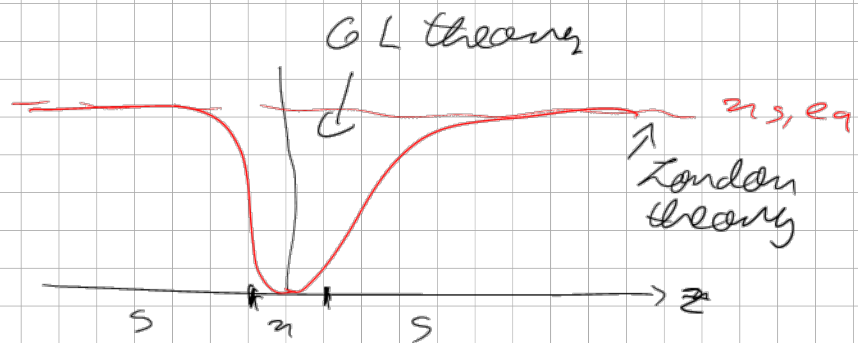
$$\xi = \sqrt{-\frac{\hbar^2}{2m_s\alpha}} \sim \underbrace{(T_c - T)^{-1/2}}_{\text{critical exponent}}$$

$$\alpha = \alpha_c'(T - T_c)$$



Flux quantum model in 1D





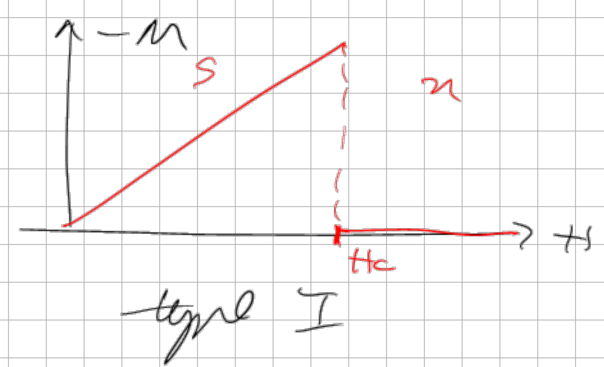
6 Critical Fields:

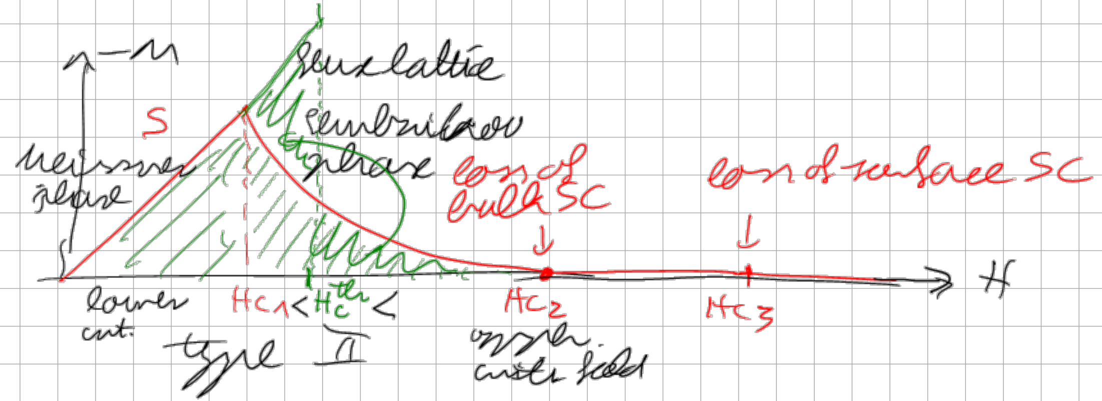
- Both type I and type II SC
- Ginzburg–Landau $\kappa = \frac{\lambda_L}{\xi} \Rightarrow \frac{1}{\sqrt{2}} = \kappa_c$

$\kappa < \kappa_c$: type I

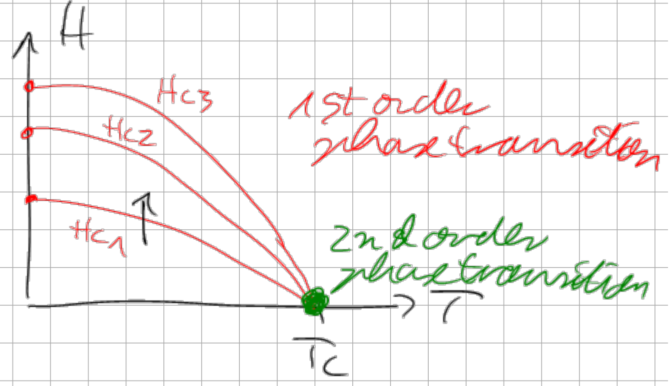
$\kappa > \kappa_c$: type II

6.1 Overview:





thermodynamic magnetic field H_c^{th} (analogous to Maxwell construction of 1st order phase transition)



green area:

$$\left.\begin{aligned} E^{II}_{mag} &= -\mu_0 V \int_0^{H_{c2}} dH\, M(H) \\ \overset{!}{=} E^{I}_{mag} &= \frac{\mu_0}{2} V H_c^{th\,2} \end{aligned}\right\} \text{ defines } \underline{H_c^{th}}$$

1) idea for H_{c2}: solve GL eqs. for small ψ $\longrightarrow$ $\underline{H_{c2}}$

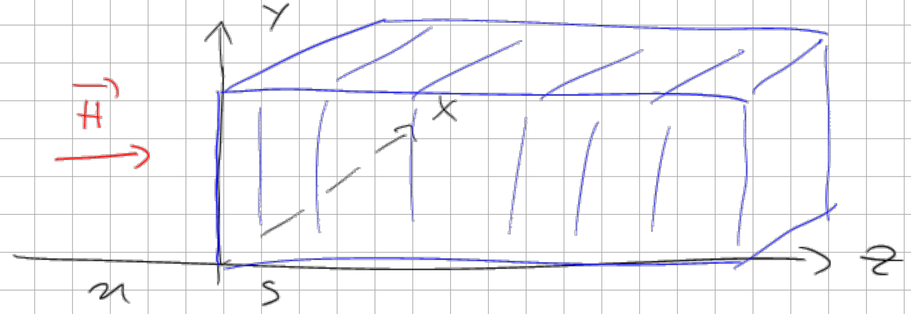
2) idea for H_{c1}: $G_{Meissner}(H_{c1}) = G_{Schubnikov}(H_{c1})$ $\longrightarrow$ defines $\underline{H_{c1}}$

6.2. Upper Critical Field H_{c2}: linearization in ψ

$$\frac{1}{2m_s}\left(-i\hbar\vec{\nabla} - e_s\vec{A}\right)^2 \psi = \alpha\psi - \cancel{\beta|\psi|^2\psi} \qquad \text{rot}\,\vec{B} = \mu_0 \vec{j}_s$$

$$\vec{j}_s = \vec{0} = \frac{i\hbar e_s}{2m_s}\left(\cancel{\psi\vec{\nabla}\psi^*} - \cancel{\psi^*\vec{\nabla}\psi}\right) - \frac{e_s^2}{2m_s}\cancel{\vec{A}|\psi|^2} \qquad \vec{B} = \text{rot}\,\vec{A}$$

6.2.1 Assumptions:



1) $\Psi \ll \Psi_0 = \sqrt{-\frac{\alpha}{\beta}}$

$\Rightarrow \vec{j}_s = \vec{0}$

$\vec{B}(\vec{r}) = \mu_0 \vec{H}(\vec{r}) = B_z \vec{e}_z, \quad B_z = \mu_0 H_z$

2) Choice of vector potential $\vec{A}$: $\quad \vec{B} = \operatorname{rot} \vec{A}$

Coulomb gauge: $\quad \operatorname{div} \vec{A} = 0$

$$\Rightarrow \vec{A}(\vec{r}) = B_z \begin{pmatrix} 0 \\ x \\ 0 \end{pmatrix}$$

$$\Rightarrow \frac{1}{2m_s}\left(-i\hbar \vec{\nabla} - e_s \vec{A}\right)^2 \Psi = -\alpha \Psi$$

6.2.2 Eigenvalue Problem:

$$-\frac{\hbar^2}{2m_s}\Delta \Psi + \frac{i\hbar e_s}{m_s} B_z x \frac{\partial \Psi}{\partial y} + \frac{e_s^2}{2m_s} B_z^2 x^2 \Psi = -\alpha \Psi$$

ansatz: $\quad \Psi(x,y,z) = \underbrace{e^{i(\gamma y + \delta z)}}_{\text{plane wave in } y \text{ and } z} \varphi(x)$

$$-\frac{\hbar^2}{2m_s}\frac{d^2\varphi(x)}{dx^2} - \frac{\hbar e_s}{m_s} B_z \gamma x \varphi(x) + \frac{e_s^2 B_z^2}{2m_s} x^2 \varphi(x) = -\left[\alpha + \frac{\hbar^2}{2m_s^2}(\gamma^2 + \delta^2)\right]\varphi(x)$$

$\Rightarrow \omega_0 = \frac{e_s B_z}{m_s}$ cyclotron frequency

$$-\frac{\hbar^2}{2m_s}\frac{d^2\varphi(x)}{dx^2} + \cancel{m_s}\overbrace{\frac{e_s^2 B_z^2}{2m_s^2}}^{=\omega_0^2}\Big(\underbrace{x - \frac{\hbar\gamma}{e_s B_z}}_{=x'}\Big)^2 \varphi(x) = -\Big(\underbrace{\alpha + \frac{\hbar^2\delta^2}{2m_s}}_{=\alpha'}\Big)\varphi(x)$$

effective 1D harmonic oscillator

$$\left(-\frac{\hbar^2}{2m_s}\frac{d^2}{dx'^2}+\frac{1}{2}m_s\omega_0^2x'^2\right)\psi(x')=-\alpha'\,\psi(x')$$

$$-\alpha_n'=\hbar\omega_0\left(n+\frac{1}{2}\right)$$ $\longrightarrow$ upper critical field

$$\psi_n(x')=N_nH_n\left(\sqrt{\frac{m_s\omega_0}{\hbar}}\,x'\right)e^{-\frac{m_s\omega_0}{\hbar}x'^2}$$ $\longrightarrow$ emerging flux lattice

Upper critical field:

$$-\left(\alpha+\frac{\hbar^2\delta^2}{2m_s}\right)=\hbar\frac{e_sB_z}{m_s}\left(n+\frac{1}{2}\right)$$

$$B_z(n,\delta)=\frac{2m_s}{\hbar e_s}\left(-\alpha-\frac{\hbar^2\delta^2}{2m_s}\right)\frac{1}{2n+1}$$

$$B_{c2}=\max_{\substack{n\in\mathbb{N}_0\\ \delta\in\mathbb{R}}}B_z(n,\delta)=B_z(0,0)=\frac{2m_s}{\hbar e_s}(-\alpha)\qquad \alpha=\alpha_c'(T-T_c)$$