5.3 Inhomogeneous Superconductors: type II SC

- no temporal variations, steady state
- now: magnetic quantities $\vec{B}$, $\vec{j}_s$ → affect the enthalpy

→ flux quantization

5.3.1 Magnetic Energy:

Magnetic induction: $\vec{B} = \mu_0 (\underbrace{\vec{H}} + \underbrace{\boxed{\vec{M}}})$

magnetic field imposed externally → external currents not described here

magnetization response of SC → origin is $\vec{j}_s$

$$\Rightarrow \quad g_1 = \frac{1}{2\mu_0} (\vec{B} - \mu_0 \vec{H})^2 = \frac{G}{V}$$

5.3.2 Kinetic Energy:

kinetic energy of superconducting electrons: $E_{ke} = \frac{1}{2} m_s \vec{v}_s^{\,2}$

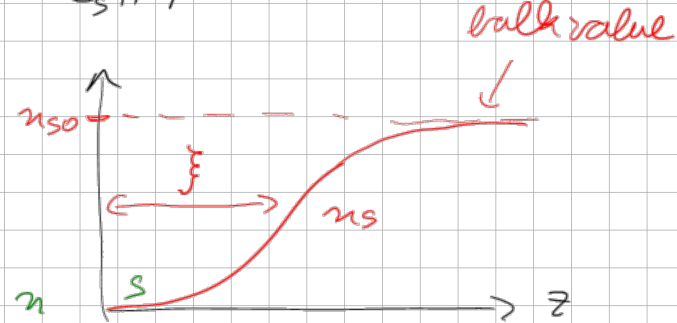
momentum: $\vec{p}_s = m \vec{v}_s + e_s \vec{A} \quad \Rightarrow \quad E_{ke} = \frac{1}{2m_s} (\vec{p}_s - e_s \vec{A})^2$

quantum mechanical description: $n_s = \Psi^* \Psi$

$$g_2 = \frac{1}{2m_s} \left| (\hat{\vec{p}}_s - e_s \vec{A}) \Psi \right|^2 , \quad \hat{\vec{p}}_s = \frac{\hbar}{i} \vec{\nabla}$$

Jordan rule

gradient of Ψ



5.3.3 Investigation of Kinetic Energy:

$$g_2 = \frac{1}{2m_s}\left[(i\hbar\vec{\nabla} - e\vec{A})\Psi^*\right]\cdot\left[(-i\hbar\vec{\nabla} - e_s\vec{A})\Psi\right]$$

go beyond London theory

$$= \underbrace{\frac{\hbar^2}{2m_s}|\vec{\nabla}\Psi|^2}_{①} - \underbrace{\frac{\hbar}{2m_s i}\underbrace{(\Psi^*\vec{\nabla}\Psi - \Psi\vec{\nabla}\Psi^*)}_{=\vec{j}_s}\cdot\vec{A}}_{②} - \underbrace{\frac{e_s^2}{2m_s}\vec{A}^2\underbrace{|\Psi|^2}_{=n_s}}_{=\frac{m_s}{2}\underbrace{\vec{v}_s^2}_{\text{velocity term}}n_s}$$

London theory

$\vec{v}_s = e_s/m_s\,\vec{A}$

① spatial inhomogeneities of order parameter Ψ

② coupling of superconducting current to vector potential

flux quantum in Chapter 3: $\vec{j}_s\cdot\vec{A} > 0 \Rightarrow$ energy gain due to $\vec{j}_s$ in $\vec{A}$

③ kinetic energy without spatial variations of Ψ

5.3.4 Free Enthalpy:

$$G_s = \int dV\, g_s = G_n + \int dV\Big\{\underbrace{\alpha|\Psi|^2 + \frac{\beta}{2}|\Psi|^4}_{\text{Landau theory}} + \underbrace{\frac{1}{2\mu_0}(\vec{B} - \mu_0\vec{A})^2}_{\text{magnetic response (London theory)}}$$

$$\boxed{\frac{1}{2m_s}\left[(+i\hbar\vec{\nabla} - e_s\vec{A})\Psi^*\right]\left[(-i\hbar\vec{\nabla} - e_s\vec{A})\Psi\right]}$$

$$+ \boxed{\underbrace{\frac{\hbar^2}{2m_s}|\vec{\nabla}\Psi|^2 - \vec{j}_s\cdot\vec{A}}_{\text{Ginzburg terms}} - \underbrace{\frac{e_s^2}{2m_s}\vec{A}^2|\Psi|^2}_{\text{London theory}}}\Big\}$$ energy balance

$\Rightarrow G_s = G_s[\Psi^*, \Psi, \vec{A}] \Rightarrow$ extremization yields Ginzburg-Landau equations

5.3.5 Variation with respect to order parameter:

volume + surface terms!

$$\delta G_s = \int_V dV\left\{(\alpha + \beta|\Psi|^2)\Psi\,\delta\Psi^* + \frac{1}{2m_s}\underbrace{(-i\hbar\vec{\nabla} - e_s\vec{A})\Psi}_{=\vec{p}_s}\cdot(i\hbar\vec{\nabla}\,\delta\Psi^* - e_s\vec{A}\,\delta\Psi^*)\right\}$$

$$\int dV \, \frac{\vec{p}_s}{2m_s} \, i\hbar \vec{\nabla} \, \delta\psi^* = \oint_{\partial V} \frac{i\hbar}{2m_s} \, \vec{p}_s \, \delta\psi^* \cdot d\vec{F}$$

$$= -\int dV \left\{ \mathrm{div}\, \vec{p}_s \, \frac{i\hbar}{2m_s} + \frac{e_s \vec{A}}{2m_s} \cdot \vec{p}_s \right\} \delta\psi^*$$

$$\delta G_s = \oint_{\partial V} \frac{-i\hbar}{2m_s} \, \delta\psi^* (i\hbar \vec{\nabla} + e_s \vec{A}) \psi \cdot \overbrace{d\vec{F}}^{= \vec{n} \cdot dF} + \int_V dV \left\{ (\alpha + \beta |\psi|^2)\psi \right.$$

$$\left. + \, \delta\psi^* \, (-i\hbar \vec{\nabla} - e_s \vec{A})^2 \frac{\psi}{2m_s} \right\}$$

volume: $\alpha\psi + \beta|\psi|^2\psi + \frac{1}{2m_s}(-i\hbar\vec{\nabla} - e_s\vec{A})^2\psi = 0$ in V (*)

surface: $\vec{n} \cdot (i\hbar\vec{\nabla} + e_s\vec{A})\psi = 0$ at ∂V

$\rightarrow$ complex conjugate due to variation with respect to ψ

5.3.6 Volume condition:

$$\hat{H}\psi = -\alpha\psi \qquad \text{GL-eq.}$$

- looks formally like a eigenvalue problem for α
- this is NOT true, as $\hat{H}$ contains a nonlinear term $\beta|\psi|^2$
- conversely: α is a material parameter, i.e. it is given

$\rightarrow$ ψ, ψ^*, $\vec{A}$ have to adjust themselves

5.3.7 Surface condition:

$\vec{n} \cdot (-i\hbar\vec{\nabla} + e_s\vec{A})\psi = 0$ (1)

$$\vec{n}\cdot(i\hbar\vec{\nabla} + e_s\vec{A})\,\psi^* = 0 \qquad (2)$$

$$-\frac{e_s}{2m_s}\left\{\psi^*(1) + \psi(2)\right\} = \Big[\underbrace{\frac{i\hbar e_s}{2m_s}(\psi\vec{\nabla}\psi^* - \psi\vec{\nabla}\psi^*) - \frac{e_s^2}{m_s}\vec{A}|\psi|^2}_{=\vec{j}_s}\Big]\cdot\vec{n} = 0$$

$\Rightarrow\ \vec{j}_s\cdot\vec{n} = 0 \ \Rightarrow$ superconducting electrons are not allowed to flow outside of the SC

5.3.8 Variation with respect to vector potential:

- ignore here Landau term
- $\vec{B} = \operatorname{rot}\vec{A} \quad (\operatorname{div}\vec{B} = 0)$

$$\delta G_s = \int_V dV\Big\{-\frac{e_s}{2m_s}\Big[\psi(i\hbar\vec{\nabla}\psi^* - e_s\vec{A}\psi)^* + \psi^*\big(-i\hbar\vec{\nabla}\psi - e_s\vec{A}\psi\big)\Big]\,\delta\vec{A}$$

$$+\frac{1}{\mu_0}\underbrace{(\vec{B}-\mu_0\vec{H})}_{\vec{u}}\operatorname{rot}\underbrace{\delta\vec{A}}_{=\vec{v}}\Big\}$$

$$\operatorname{div}(\vec{u}\times\vec{v}) = \vec{v}\cdot\operatorname{rot}\vec{u} - \vec{u}\operatorname{rot}\vec{v}$$

$$= -\frac{1}{\mu_0}\int_V \operatorname{div}\big((\vec{B}-\mu_0\vec{H})\times\delta\vec{A}\big) + \int_V dV\,\delta\vec{A}\,\frac{1}{\mu_0}\underbrace{\operatorname{rot}(\vec{B}-\mu_0\vec{H})}_{=\mu_0\vec{j}_s} \qquad (***)$$

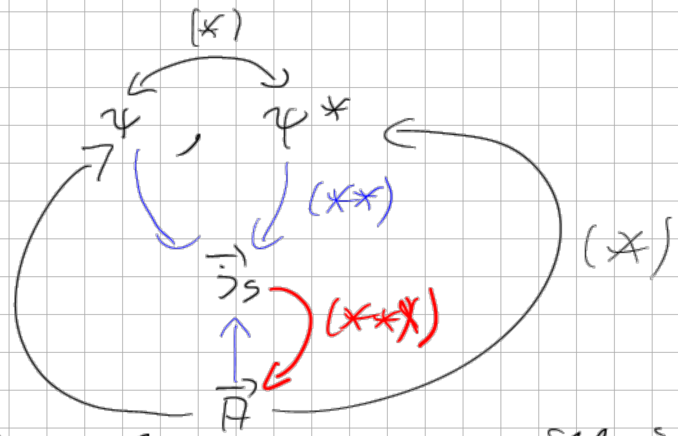
$$= -\oint_{\partial V}\frac{1}{\mu_0}\big[(\vec{B}-\mu_0\vec{H})\times\delta\vec{A}\big]\cdot\overbrace{d\vec{F}}^{=\vec{n}\,dF} + \int_V dV\,\vec{j}_s\cdot\delta\vec{A}$$

volume: $\vec{j}_s = \frac{i\hbar e_s}{2m_s}(\psi\vec{\nabla}\psi - \psi^*\vec{\nabla}\psi) - \frac{e_s^2}{m_s}|\psi|^2\vec{A}$ recovered in V (**)

surface: $(\vec{B}-\mu_0\vec{H})\times\vec{n} = \vec{0}$ at ∂V $\Rightarrow$ $\vec{B}-\mu_0\vec{H}\big|_{\text{tang.}}$ is continuous at ∂V

5.3.9 Summary:

nonlinear coupled set of eqs. for $\psi, \psi^*, \vec{j}_s, \vec{A}$



- so complicated that no analytic solution is possible in general
  $\rightarrow$ numerical solution needed
- under certain physical conditions simplifying assumptions possible $\rightarrow$ approximate solutions

5.4 Characteristic Length Scales:

$\varkappa = \frac{\lambda_L}{\xi}$

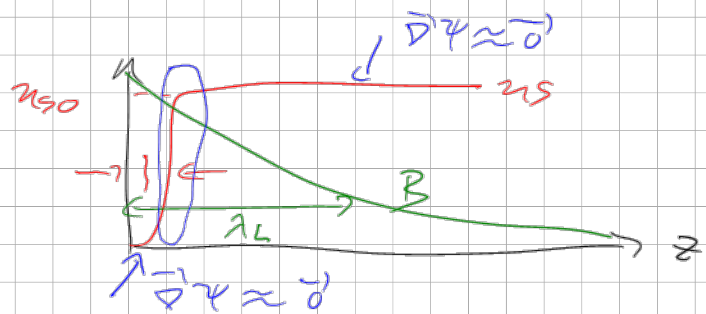
- London penetration length λ_L
- Coherence length ξ

$\left.\begin{array}{l} \text{London penetration length } \lambda_L \\ \text{Coherence length } \xi \end{array}\right\} \left\{\begin{array}{l} \text{type II SC: } \varkappa > 1, \ \lambda_L > \xi \\ \text{type I SC: } \varkappa < 1, \ \lambda_L < \xi \end{array}\right.$

5.4.1 Type II:

high T_C SC: $\lambda_L = 1500$ Å, $\xi = 10$ Å

approximation: $\vec{\nabla}\psi \approx \vec{0}$



$$\text{rot}\,\vec{j}_s = -\frac{e_s^2 n_s}{m_s}\,\text{rot}\,\vec{A} = -\frac{e_s^2 n_s}{m_s}\,\vec{B}$$

$$\text{rot}\,\vec{B} = \mu_0 \vec{j}_s, \quad \text{rot rot}\,\vec{B} = -\Delta\vec{B} = \mu_0\,\text{rot}\,\vec{j}_s = -\frac{e_s^2 n_s}{m_s}\mu_0 \vec{B} \Rightarrow$$ Helmholtz equation

$\Rightarrow$ GL-theory reduces approximately to London theory

$$\lambda_L = \sqrt{\frac{m_s}{e_s^2 \mu_0 |\Psi(T)|^2}} \underset{\substack{\uparrow \\ \text{Landau theory}}}{\sim} \frac{1}{\sqrt{T_c - T}}$$ diverges at T_c

5.4.2 Extreme SC of type I $\qquad \xi \sim \frac{1}{\sqrt{T_c - T}}$ diverges at T_c

Topics:

1) BCS theory

2) Josephson effect

3) Gorkov derivation of GL theory from BCS theory

4) High T_c superconductors

5) Nobel prize 2025: macroscopic tunneling
 TODAY, 19.00: 46-215 by A. Kamra

6) strange surface SC: topology

Procedure:

- Drop an email until 31.12.25
- Plan for talks beginning of January
- Talks scheduled on 21.1.26