4. London Equations:

- superconductor: ideal conductor + ideal diamagnet
  (Meissner-Ochsenfeld effect (1933))
- London theory (1935) = Maxwell theory + matter equations for superconductor

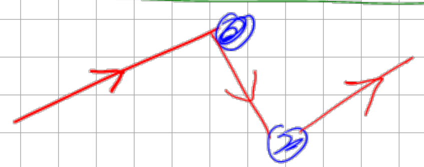
4.1 London Equations:

Landau two fluid model: $\vec{j} = \vec{j}_n + \vec{j}_s$

electrons = "n" (normal conducting) + "s" (superconducting)

4.1.1 Normal conducting electrons:

- metal: electrons scatter at atomic cores ($\vec{E}$)



  > permanent change of velocity
  > resistivity

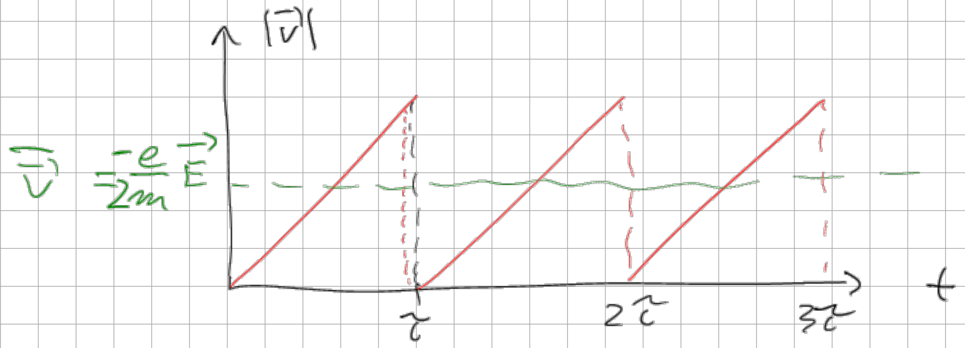
- motion is affected in two ways: drift and diffusion
  > diffusion
  thermal average: $\overline{\vec{s}} = \vec{0}$, $\overline{\vec{v}} = \vec{0}$, $\overline{\vec{a}} = \vec{0}$
  > drift: due to presence of electric field → Drude model

$$m\vec{a} = -e\vec{E}, \quad \vec{a} = -\frac{e}{m}\underbrace{\vec{E}}_{\text{constant}} \quad \Rightarrow \quad \vec{v} = -\frac{e}{m}\vec{E}\underbrace{\tau}_{\substack{\text{mean free time} \\ (10-100\ \text{fs})}}$$

$$\vec{j}_n = -e\, n_n\, \overline{\vec{v}} = \underbrace{\frac{e^2 n_n \tilde{\tau}}{2m}}_{=\sigma_n} \vec{E} \quad \text{Ohm law}$$

electric conductivity $= \sigma_n$

$e = e_n\,,\quad m = m_n$



4.1.2 Superconducting Electrons:

- no scatterings, no electric resistivity
- Cooper pairs: $e_s = 2e,\quad m_s = 2m$
- Electric field: Newton law:

$$\left.\begin{aligned} m_s \frac{d}{dt}\vec{v}_s &= -e_s \vec{E} \\ \vec{j}_s &= -e_s n_s \vec{v}_s \end{aligned}\right\} \quad \underbrace{\frac{d}{dt}}_{= \underbrace{\frac{\partial}{\partial t}}_{\text{explicit time derivative}} + \underbrace{\vec{v}_s\cdot\vec{\nabla}}_{\text{transport derivative}}}\left(\frac{m_s}{e_s^2 n_s}\vec{j}_s^{\,(\vec{x},t)}\right) = \vec{E} \quad \Bigg\}\; \left(\frac{\partial}{\partial t} - \underbrace{\cancel{\frac{\vec{j}_s}{e_s n_s}\cdot\vec{\nabla}}}_{\text{neglection}}\right)\left(\frac{m_s}{e_s^2 n_s}\vec{j}_s\right) = \vec{E}$$

nonlinearity

First London equation: $\frac{\partial}{\partial t}(\Lambda_s \vec{j}_s) = \vec{E}\,,\quad \Lambda_s = \frac{m_s}{e_s^2 n_s}$

4.1.3 Remarks:

normal conducting e^-	superconducting e^-
$\vec{j}_n = \sigma_n \vec{E}$ (Ohm law)	$\frac{\partial}{\partial t}(\Lambda_s \vec{j}_s) = \vec{E}$ 1st London equation
$\vec{E} = \vec{0} \Rightarrow \vec{j}_n = \vec{0}$	$\vec{E} = \vec{0} \Rightarrow \vec{j}_s = \text{const.}$ (can be non-zero in time)

4.1.4 Induction Law:

$$\mathrm{rot}\,\vec{E} = \underset{\substack{\uparrow \\ \text{Lenz rule}}}{-}\frac{\partial \vec{B}}{\partial t} \qquad \text{(local formulation)}$$

conversion to global formulation

$$\int_F \mathrm{rot}\,\vec{E}\cdot d\vec{F} \overset{\text{Stokes}}{=} \underbrace{\oint_{\partial F} \vec{E}\cdot d\vec{r}}_{U_{ind}} = -\int_F \frac{\partial \vec{B}}{\partial t}\cdot d\vec{F} \overset{\text{"}\frac{\partial}{\partial t}F\text{"}}{=} -\frac{\partial}{\partial t}\underbrace{\int_F \vec{B}\cdot d\vec{F}}_{=\Phi}$$

magnetic flux

$$\Rightarrow \quad U_{ind} = -\frac{\partial}{\partial t}\Phi \qquad \text{(global formulation)}$$

$\Rightarrow$ useful later for flux quantization.

4.1.5 Second London equation:

$$\frac{\partial}{\partial t}(\Lambda_s \vec{j}_s) = \vec{E} \quad \Big|\ \mathrm{rot} \quad \Rightarrow \frac{\partial}{\partial t}\mathrm{rot}(\Lambda_s \vec{j}_s) = \mathrm{rot}\,\vec{E} \overset{\text{induction law}}{=} -\frac{\partial \vec{B}}{\partial t}$$

$$\Rightarrow \frac{\partial}{\partial t}\left\{\mathrm{rot}(\Lambda_s \vec{j}_s) + \vec{B}\right\} = \vec{0} \quad \Rightarrow \quad \left\{\mathrm{rot}(\Lambda_s \vec{j}_s) + \vec{B} = \text{const.} = ?\right.$$

volume of superconductor:

- $\Lambda_s = \frac{m_s}{e_s^2 n_s}$ $\quad n_s = \text{const in London theory} \Rightarrow \Lambda_s = \text{const.}$
- $\vec{E} = \vec{0} \Rightarrow \vec{j}_s = \text{const.} \quad \Rightarrow \mathrm{rot}(\Lambda_s \vec{j}_s) = \vec{0}$
- Meissner Ochsenfeld effect: $\vec{B} = \vec{0}$

$\Rightarrow ? = 0$

$$\Rightarrow \quad \mathrm{rot}(\Lambda_s \vec{j}_s) + \vec{B} = \vec{0} \qquad \text{2nd London equation}$$

4.1.6 Conclusions:

$\operatorname{div}\vec{B} = 0$ (Maxwell eq.) + Helmholtz theorem: $\vec{B} = \operatorname{rot}\vec{A}$

$$\xrightarrow[\text{Lond. eq.}]{\text{2nd}} \quad \operatorname{rot}(\Lambda_s \vec{j}_s + \underbrace{\vec{A}}_{\text{gauge freedom: add. grad. field}}) = \vec{0} \Rightarrow \vec{A} = -\Lambda_s \vec{j}_s \Rightarrow \vec{j}_s = -\frac{e_s^2 n_s}{m_s} \cdot \vec{A} = -e_s n_s \vec{v}_s$$

$$\Rightarrow \vec{v}_s = \underbrace{\frac{e_s}{m_s}}_{= \frac{e}{m} \text{ specific charge}} \vec{A}$$

4.2 Field Equation for a Superconductor:

$\vec{E}, \vec{B} \quad + \quad \rho, \vec{j}$

4.2.1 System of Equations:

$$\operatorname{div}\vec{E} = \frac{\rho}{\varepsilon_0} \quad (M1) \qquad \operatorname{rot}\vec{B} = \mu_0 \vec{j} + \underbrace{\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}}_{\text{Maxwell displacement current}} \quad (M3)$$

$$\operatorname{rot}\vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (M3) \qquad \operatorname{div}\vec{B} = 0 \quad (M4)$$

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \quad (M5), \qquad \text{superconductor: } \varepsilon_r \approx 1, \ \mu_r \approx 1$$

$$\vec{j} = \vec{j}_n + \vec{j}_s \quad (L3) \qquad \frac{\partial}{\partial t}(\Lambda_s \vec{j}_s) = \vec{E} \quad (L1)$$

$$\vec{j}_n = \sigma_n \vec{E} \quad (L4) \qquad \operatorname{rot}(\Lambda_s \vec{j}_s) = -\vec{B} \quad (L2)$$

4.2.2 Consistency Relations:

1) $\frac{\partial}{\partial t}\rho \overset{(M1)}{=} \varepsilon_0 \,\mathrm{div}\,\frac{\partial \vec{E}}{\partial t} \overset{(M3)}{=} -\varepsilon_0 \mu_0 c^2 \,\mathrm{div}\,\vec{j} \overset{(M5)}{=} -\mathrm{div}\,\vec{j}$ (M6)

2) $\mathrm{rot}\,\vec{E} \overset{(L1)}{=} \frac{\partial}{\partial t}\,\mathrm{rot}\,(\Lambda_s \vec{j}_s) \overset{(L2)}{=} -\frac{\partial \vec{B}}{\partial t} \;\hat{=}\; (M3)$

3) $\mathrm{div}\,\vec{B} \overset{(L2)}{=} 0 \;\hat{=}\; (M4)$

London equation (L1), (L2) substitute homogeneous Maxwell equations (M3), (M4) for a superconductor.

4.2.3 Elimination of charge and current densities:

Determine one equation for $\vec{E}$ and $\vec{B}$, respectively

- (L3), (L4): eliminate $\vec{j}_n \longrightarrow \vec{j}_s = \vec{j} - \sigma_n \vec{E}$ $(*)$
- eliminate $\vec{j}_s$:

(L1) and $(*)$: $\vec{E} + \frac{\partial}{\partial t}\left(\Lambda_s \sigma_n \vec{E}\right) = \frac{\partial}{\partial t}\left(\Lambda_s \vec{j}\right)$ (L1)'

(L2) and $(*)$ and (M3): $\vec{B} + \frac{\partial}{\partial t}\left(\Lambda_s \sigma_n \vec{B}\right) = -\mathrm{rot}\left(\Lambda_s \vec{j}\right)$ (L2)'

Equation for $\vec{B}$:

$$\mathrm{rot}\,\vec{B} = \mu_0 \vec{j} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \quad (M2) \quad \Big|\ \mathrm{rot}$$

$$\underbrace{\mathrm{rot}\,\mathrm{rot}}_{=\mathrm{grad}\,\overset{(M4)}{\cancel{\mathrm{div}}}\,-\Delta}\,\vec{B} = \mu_0\,\mathrm{rot}\,\vec{j} + \frac{1}{c^2}\frac{\partial}{\partial t}\underbrace{\mathrm{rot}\,\vec{E}}_{\overset{(M3)}{=}-\frac{\partial}{\partial t}\vec{B}}$$

$$\underbrace{\underbrace{\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{B} - \Delta\vec{B}}_{\text{"wave equation"}} + \underbrace{\boxed{\mu_0\sigma_n\frac{\partial\vec{B}}{\partial t}}}_{\text{normal cond. e}}}_{\text{telegraph equation}} + \underbrace{\frac{\mu_0}{\Lambda_s}}_{\frac{1}{\text{"length"}^2}}\vec{B} = \vec{0}$$

homogeneous extension of telegraph equation

supercond. e

skin effect

magnetic field decreases along

$$d(\omega) = \sqrt{\frac{2}{\mu_0\sigma_n\omega}}$$ penetration length

London length:

$$\lambda_L = \sqrt{\frac{\Lambda_s}{\mu_0}} = \sqrt{\frac{m_s}{e_s^2 n_s\mu_0}}$$

Equation for electric fields:

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} - \Delta\vec{E} + \mu_0\sigma_n\frac{\partial\vec{E}}{\partial t} + \frac{\mu_0}{\Lambda_s}\vec{E} = -\frac{1}{\varepsilon_0}\operatorname{grad}\varrho$$

inhomogeneous extension of telegraph equation