London theory = Maxwell theory

+ matter equations of a superconductor

$$\vec{j} = \vec{j}_n + \vec{j}_s$$

Drude model

$$\vec{j}_n = \sigma_n \vec{E}$$

$$\sigma_n = \frac{e^2 n_n \tau}{2m}$$

$$\frac{\partial}{\partial t}(\Lambda_s \vec{j}_s) = \vec{E} \quad (L1)$$

$$\mathrm{rot}(\Lambda_s \vec{j}_s) = -\vec{B} \quad (L2)$$

$$\Lambda_s = \frac{m_s}{e_s^2 n_s}$$

Consequence:

$$\underbrace{\underbrace{\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{B} - \Delta \vec{B}}_{\text{wave equation}} + \underbrace{\mu_0 \sigma_n \frac{\partial \vec{B}}{\partial t}}_{\text{skin effect}}}_{\text{telegraph equation}} + \underbrace{\frac{\mu_0}{\Lambda_s}}_{\text{London penetration length}} \vec{B} = \vec{0}$$

$$\lambda(\omega) = \sqrt{\frac{2}{\mu_0 \sigma_n \omega}}$$

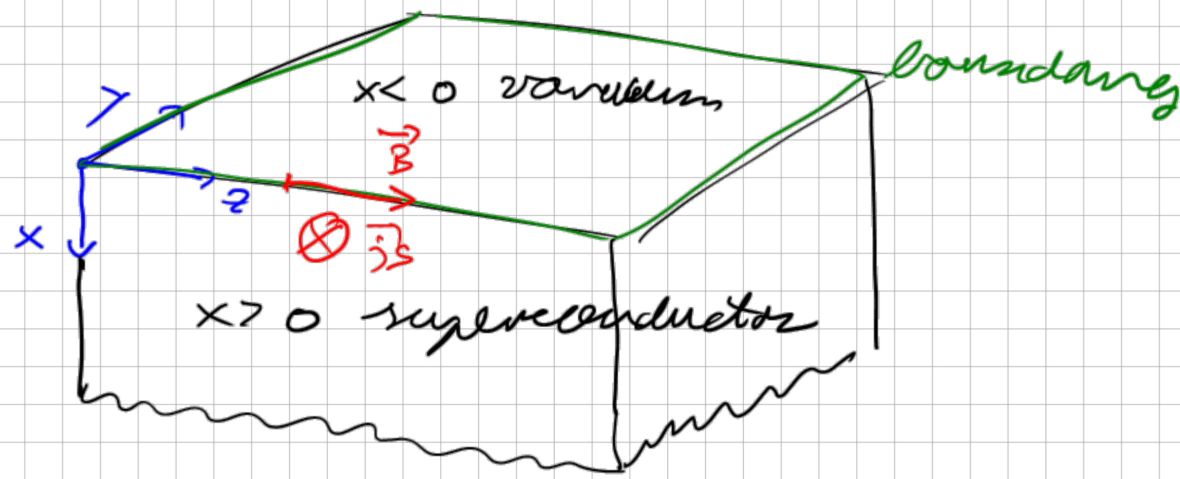
$$\lambda_L = \sqrt{\frac{\Lambda_s}{\mu_0}} = \sqrt{\frac{m_s}{\mu_0 e_s^2 n_s}}$$

4.3 Stationary Case:

$$\vec{E} = \vec{0}, \quad \rho = 0, \quad \frac{\partial \vec{B}}{\partial t} = \vec{0}, \quad \frac{\partial \vec{j}_s}{\partial t} = \vec{0}$$

Helmholtz equation: $\Delta \vec{B} - \frac{1}{\lambda_L^2}\vec{B} = \vec{0}$, $\Delta \vec{j}_s - \frac{1}{\lambda_L^2}\vec{j}_s = \vec{0}$

Geometry of a superconducting half space



1) $\vec{B} = \vec{B}(x)$, $\vec{j}_s = \vec{j}_s(x)$

2) $\mathrm{rot}\,\vec{B} = \mu_0 \vec{j}_s \Rightarrow \begin{pmatrix} 0 \\ -\frac{\partial B_z}{\partial x} \\ +\frac{\partial B_y}{\partial x} \end{pmatrix} = \mu_0 \begin{pmatrix} j_{sx} \\ j_{sy} \\ j_{sz} \end{pmatrix}$

3) $\vec{B}(x) = B_z(x)\,\vec{e}_z$ $\qquad \vec{j}_s(x) = j_{sy}\,\vec{e}_y$

$\downarrow$ Helmholtz equation

$\vec{B}(x) = B_{z0}\, e^{-\frac{x}{\lambda_L}}\,\vec{e}_z \longrightarrow \vec{j}_s(x) = \frac{\mu_0}{\lambda_L}\, e^{-\frac{x}{\lambda_L}}\,\vec{e}_y$; $x \geq 0$

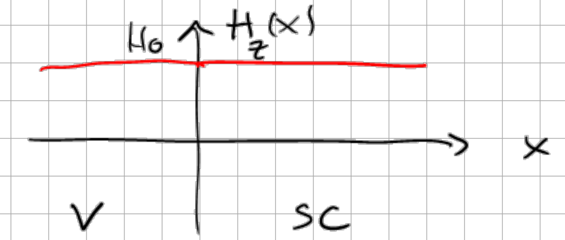
$x \geq 0$

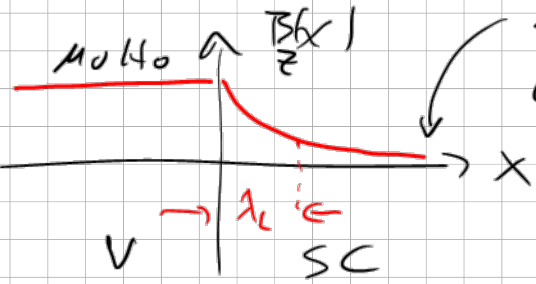
$\vec{B}(x) = B_{z0}\,\vec{e}_z \quad x \leq 0$

$\vec{B}(x) = \mu_0 \vec{H}(x) \quad x \leq 0$

$$\vec{H}(x) = H_0 \qquad x \leq 0$$

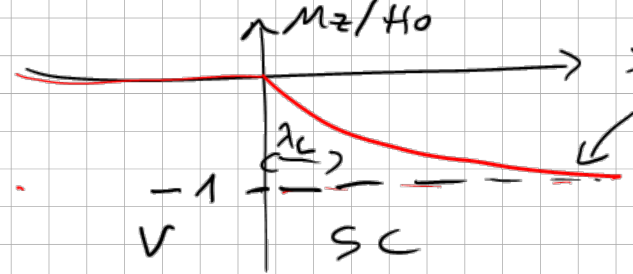
$$\Rightarrow B_{z0} = H_0 \mu_0$$





Meissner–Ochsenfeld effect

no currents



ideal diamagnet

$$\vec{M}(x) = \frac{\vec{B}(x)}{\mu_0} - \vec{H}(x) = H_0\left(e^{-\frac{x}{\lambda_L}} - 1\right)\vec{e}_z$$

London: quantitative description of Meissner–Ochsenfeld effect

4.4 Superconducting Electrons:

Assume: $e_s = e$, $m_s = m$

$$\lambda_L = \sqrt{\frac{m}{\mu_0 e^2 n_s'}} \Rightarrow n_s' = \frac{m}{\mu_0 e^2} \cdot \frac{1}{\lambda_L^2} = 2.8 \cdot 10^{13} \frac{1}{m} \cdot \frac{1}{\lambda_L^2}$$

element	Al	Cd	In	Pb (lead)
$\lambda_L / \mathring{A}$	500	1300	640	390
$n_s' / 1/m^3$	$1.1\ 10^{28}$	$1.7\ 10^{27}$	$6.8 \cdot 10^{27}$	$1.8\ 10^{28}$

density of normal conducting electrons

assumption: 1 electron per atom

$$n = \frac{N}{V} = \frac{M/V}{M/N} = \frac{\rho_m}{m_{at}}$$

ρ_m ← massdensity ; m_{at} ← atomic mass

total number of electrondensity

element	Al	Cd	In	Pb
n / $1/m^3$	$6 \cdot 10^{28}$	$4.6 \; 10^{28}$	$3.8 \; 10^{28}$	$3.3 \; 10^{28}$

$n_n = n - n_s'$: normal conducting electron density

$\Rightarrow \; n_n > n_s'$: only a fraction of electrons are SC !

4.5 Remarks:

1) Cooper pairs: $n_s = \frac{n_s'}{2}$

$$\lambda_L = \sqrt{\frac{m_s}{e_s^2 n_s \mu_0}} = \sqrt{\frac{2m}{(2e)^2 \frac{n_s'}{2} \mu_0}} = \sqrt{\frac{m}{e^2 n_s' \mu_0}}$$

2) normal conductor

$n_s \downarrow 0$

$\lambda_L \to \infty$

superconductor

n_s finite

λ_L finite

4) London penetration length λ_L: small temperature dependence

This is not the only length scale characterizing SC

BCS theory: Cooper pairs → Cooper pair size ξ 2nd length scale

5) Ginzburg-Landau theory: λ_L and ξ

$$\varkappa = \frac{\lambda_L}{\xi}$$

$\varkappa < \frac{1}{\sqrt{2}} \approx 0.7$	$\varkappa > \frac{1}{\sqrt{2}} \approx 0.7$
SC of type I	SC of type II

4.6 Conservation of Fluxoid and its Quantization:

so far: simply connected SC region

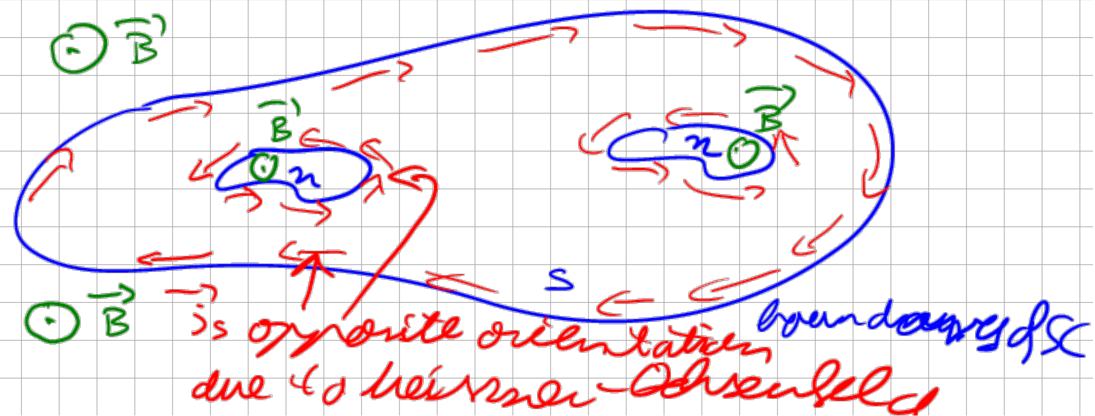
now: not a simply connected SC region

4.6.1 Conservation of Fluxoid:

SC of type II

SC and NC regions provided

$B_{c1}(T) < B < B_{c2}(T)$ and $T < T_c$



induction law (M3): $-\underbrace{\int_F \frac{\partial \vec{B}}{\partial t} \cdot d\vec{F}}_{= -\frac{\partial}{\partial t} \underbrace{\int_F \vec{B} \cdot d\vec{F}}_{= \Phi}} = \dots = \oint_{\partial F} \underbrace{\vec{E}}_{= \frac{\partial}{\partial t}(\Lambda_s \vec{j}_s)} \cdot d\vec{r}$

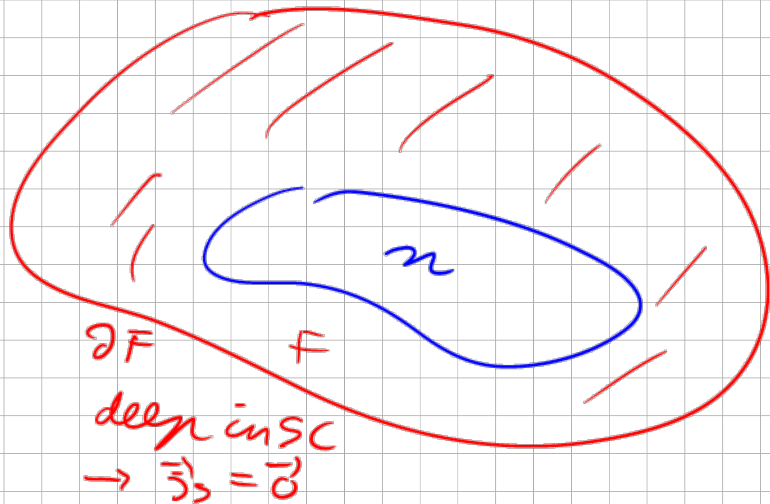
$\frac{\partial F}{\partial t}$

$$\Rightarrow \frac{\partial}{\partial t} \Big\{ \underbrace{\Phi + \Lambda_s \oint_{\partial F} \vec{j}_s \cdot d\vec{r}}_{= \phi' \,\hat{=}\, \text{fluxoid}} \Big\} = 0$$

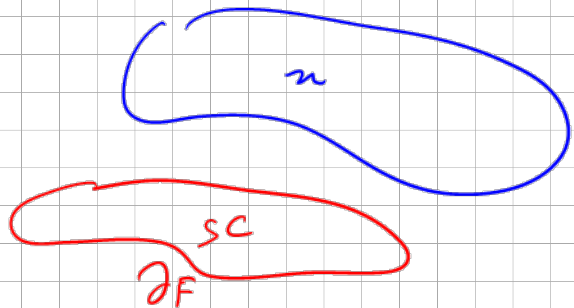
magnetic flux

superconducting current density

1) $\phi' = \phi$, ∂F deep in SC



2)



$$\phi = \int_F \underbrace{\vec{B}}_{\overset{(L2)}{=} \mathrm{rot}(\Lambda_s \vec{j}_s)} \cdot d\vec{F} \overset{\text{Stokes}}{=} -\Lambda_s \oint_{\partial F} \vec{j}_s \cdot d\vec{r} \Rightarrow \phi' = 0 \quad \text{in SC}$$

4.6.2 Reformulation:

Helmholtz

$$-\vec{B} = \mathrm{rot}\left(\Lambda_s \vec{j}_s\right) \ (L2) \longrightarrow (M4)\ \mathrm{div}\,\vec{B} = 0 \Rightarrow \vec{B} = \mathrm{rot}\,\vec{A}$$

$$\boxed{\phi'} = \underbrace{\int_F \underbrace{\vec{B}'}_{=\mathrm{rot}\,\vec{A}'} \cdot d\vec{F}'}_{\text{Stokes}} + \frac{m_s}{e_s^2 \cancel{n_s}} \oint_{\partial F} e_s \cancel{n_s}\, \vec{v}_s \cdot d\vec{r}' \Big\}$$

n_s is eliminated

$$= \oint_{\partial F} \vec{A}' \cdot d\vec{r}'$$

$$= \oint_{\partial F} \left\{ \vec{A} + \frac{m_s}{e_s} \vec{v}_s \right\} \cdot d\vec{r} = \frac{1}{e_s} \oint_{\partial F} \underbrace{\Big(\underbrace{m_s \vec{v}_s}_{=\vec{p}_s} + e_s \vec{A} \Big)}_{\text{canonical momentum } \vec{P}_s} \cdot d\vec{r}$$

$$\phi' = \frac{1}{e_s} \underbrace{\oint_{\partial F} \vec{P}_s \cdot d\vec{r}}_{\text{closed line integral}}$$

4.6.3 Quantization:

London theory is classical, but it allows for semiclassical quantization (Bohr-Sommerfeld):

$$\oint_{\partial F} \vec{P}' \cdot d\vec{r}' = n h, \quad n \in \mathbb{N}_0$$

$$\Rightarrow \phi'_n = n \frac{h}{e_s}, \quad n \in \mathbb{N}_0$$

fluxoid is quantized

unit: $\phi_0 \underset{\uparrow \atop e_s = 2e}{=} \frac{h}{2e} = 2.07 \cdot 10^{-15}\ Tm^2$

meaning quantization of $\vec{\phi}$ leads to $e_s = 2e$

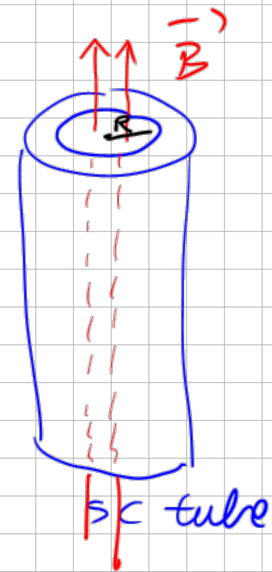
$\Rightarrow$ superconductivity is a macroscopic quantum phenomenon.

4.7 Measurement of Fluxoid Quanta:

1961: Munich Doll – Näbauer

Stanford Deaver – Fairbank

4.7.1 General Idea:



switch on $\vec{B}$ $\Rightarrow$ SC current in tube shell

$\vec{B} \to \vec{0}$: SC remain

Magnetic flux is not continuous but quantized

experimental accuracy: few flux quanta

$\to$ small magnetic induction

$B = \frac{\phi_0}{\pi R^2} \underset{\uparrow \atop R = 5\mu m}{=} 26\ \mu T$ (earth magnetic field $30\mu T$)

must be shielded